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The Study of Stress Intensity Factor for Cracked Gear Tooth by Using Weight Function Method

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ABSTRACT

For a cracked gear tooth, the stress intensity factor is integrated by weight function method. The accuracy of result depends on the selection of the stress field and weight functions. In this study the loading is resolved to a point force in x direction, a point force in y direction and a bending moment shown in Figure2. The bending stress due to the point force in y direction is considered as the constant through the cross section by Flasker. In this work, the bending stress due to the point force in y direction is determined by three approaches. For third approach, the gear tooth is simulated as one end fixed, and the stress is determined by the theory of elasticity. The stress field of third approach is more reasonable and leads to a reasonable stress intensity factor. Additionally, the weight functions used by Flasker are the complicated formulas of integration. These weight functions are simplified to the polynomial equations and the stress intensity factor becomes easy and convenient to determine.

Key words: *gear tooth, stress intensity factor, weight function method.*

I. Introduction

The gear is one of the indispensable components for power transmission of mechanism. The cyclic loading leads to an initial fatigue crack in the fillet region of the gear tooth, and the fatigue crack grows until failure of gear tooth occurrs. Based on the theory of fracture mechanics, stress intensity factor is the most important parameter for fatigue crack growth. The loading distribution and the geometrical condition of crack tip are considered as two major parameters of calculating the stress intensity factor.

Generally, stress intensity factor can be obtained by numerical methods. Finite element method is the most common approach to compute the stress intensity factor. Flasker[1] used finite element method to discuss the service life of cracked gear. Lwicki[2] also used finite element method to find the effect of rim thickness on gear crack propagation path. Flasker[3] described the influence of contact area on service life of gears with crack in tooth root by finite element method. This work requires a large number and suitable type of elements in the crack tip region and complete remeshing for each fatigue cycle while the crack length is changed. Besides finite element method, Daniewicz[4] applied the principles of fracture mechanics, conformal mapping technique and complex potential method to compute the stress intensity factor for cracked gear tooth. Flasker[5] described an approach to determine the stress intensity factor of cracked gear teeth using the weight function method.

Since the formulas of stress intensity factor in many researches are still lacking a complete description of cracked gear tooth, in this study a thorough description of stress intensity factor of cracked gear tooth is presented by modifying Flasker's stress field [5].

II. Weight Function Method

Consider an edge crack of length a located at the root of gear tooth as shown in Figure 1. The applied

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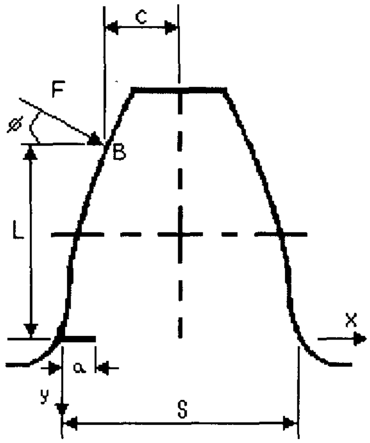

Figure 1 Model of loading on cracked gear tooth.

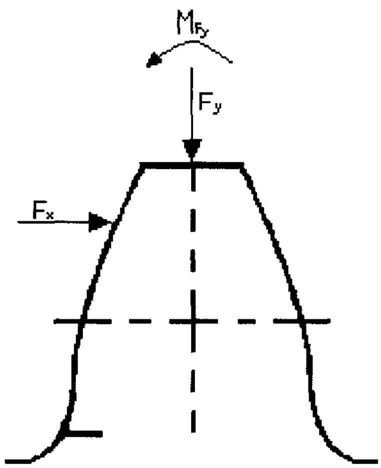

Figure 2 Equivalent loading on cracked gear tooth.

force F can be replaced by an equivalent force and moment system as shown in Figure 2. For two-dimensional crack problems, the stress intensity factor has been expressed by Abersek and Flasker[5] as the formula

$$K = \int_0^a \sigma(x) m(a,x) dx \tag{1}$$

The weight function $m(a,x)$ can be obtained from the compact test specimen as a reference solution.

$$m(a,x) = \frac{\sqrt{2}}{\sqrt{\pi(a-x)}}\left[1 + m_1\left(\frac{a-x}{a}\right) + m_2\left(\frac{a-x}{a}\right)^2\right] \tag{2}$$

where

$$m_1 = \frac{1}{Y_t(\alpha)}\left[2\alpha Y_t'(\alpha) + \frac{3}{4}G(\alpha)\right] + 1 \tag{3}$$

$$m_2 = \frac{1}{4Y_t(\alpha)}\left[2\alpha G'(\alpha) - G(\alpha)\right] \tag{4}$$

in which $\alpha = a/S$. $Y_t(\alpha)$ is referred to as the shape factor and $G(\alpha)$ is given by

$$G(\alpha) = \frac{\sqrt{a}}{I_3(a)}\left[I_1(a) - 4\sqrt{a} I_2(a) Y_t(\alpha)\right] \tag{5}$$

The quantities $I_1(a)$, $I_2(a)$ and $I_3(a)$ are

$$I_1(a) = \pi\sqrt{2}\sigma_0 \int_0^a [Y_t(\alpha)]^2 \alpha d\alpha$$

$$I_2(a) = \int_0^a \sigma_r(x)\sqrt{a-x}\,dx \quad I_3(a) = \int_0^a \sigma_r(x)(a-x)^{3/2} dx \tag{6}$$

Consider the components F_x and F_y of force F shown in Figure 2.

$$F_x = F\cos\phi \text{ and } F_y = F\sin\phi \tag{7}$$

where ϕ is the pressure angle.
Assume that F_y contributes a uniform stress.

$$\sigma_{y0} = \frac{F_y}{bS} \tag{8}$$

The stress due to the moment generated by F_x and M_{F_y} is

$$\sigma_m(x) = \frac{6(F_x L - M_{F_y})}{bS^2}\left(1 - \frac{2x}{S}\right) \tag{9}$$

where $M_{F_y} = F_y c$.

Thus $\sigma(x)$ can be thought of the combined stress as

$$\sigma(x)=\sigma_m(x)-\sigma_{y0} \tag{10}$$

The shear stress on the crack plane is

$$\tau_{xy}=\frac{F_x}{bS} \tag{11}$$

The shear stress which contributes to mode II crack is neglected in this study. Therefore, an expression for mode I stress intensity factor K of cracked gear tooth can be calculated from equation (1).

$$K=\sqrt{\frac{2}{\pi}}\sqrt{a}\int_0^a\left[\frac{6FL}{bS^2}\left(\cos\phi-\tfrac{c}{L}\sin\phi\right)\left(1-\frac{2x}{S}\right)-\sigma_{y0}\right]\times\left[1+m_1\left(\frac{a-x}{a}\right)+m_2\left(\frac{a-x}{a}\right)^2\right]\frac{dx}{\sqrt{a(a-x)}} \tag{12}$$

Equation (12) may be regarded as the combination of K_m for bending and K_{F_y} for extension, that is

$$K=K_m+K_{F_y} \tag{13}$$

K_m and K_{F_y} can be written as

$$K_m=\frac{6FL}{bS^2}\sqrt{\pi a}\left(\cos\phi-\frac{c}{L}\sin\phi\right)Y_m(\alpha) \tag{14}$$

$$K_{F_y}=-\frac{F_y}{bS}\sqrt{\pi a}\,Y_t(\alpha) \tag{15}$$

$Y_m(\alpha)$ and $Y_t(\alpha)$ are obtained from equation (14) and equation (15).

$$Y_m(\alpha)=\frac{\sqrt{2}}{\pi}\int_0^a\left(1-\frac{2x}{S}\right)\left[1+m_1\left(\frac{a-x}{a}\right)+m_2\left(\frac{a-x}{a}\right)^2\right]\frac{dx}{\sqrt{a(a-x)}} \tag{16}$$

$$Y_t(\alpha)=\frac{\sqrt{2}}{\pi}\int_0^a\left[1+m_1\left(\frac{a-x}{a}\right)+m_2\left(\frac{a-x}{a}\right)^2\right]\frac{dx}{\sqrt{a(a-x)}} \tag{17}$$

Note that $Y_t(\alpha)$ for the compact tension specimen can be written as

$$Y_t(\alpha)=1.1215-0.231\alpha+10.55\alpha^2-21.72\alpha^3+30.39\alpha^4 \tag{18}$$

III. Three Approaches

In Figure 2, moment M_{F_y} generated by F_y is considered as a constant $M_{F_y}=F_y c$ for cracked gear tooth by Flasker [5]. In reality, F_y is a point load, and may affect $\sigma_m(x)$. In order to simulate the stress and stress intensity factor more accurately, three approaches are presented to find the stress $\sigma_m^{F_y}(x)$ and stress intensity factor $K_m^{F_y}$ generated by M'_{F_y} and compared with the results.

A. First Approach

According to the point of view of solid mechanics, the moment that is a function of position can be shown as

$$M'_{F_y}=-F_y\left(x-\frac{S}{2}\right) \tag{19}$$

Similar in equation (9), $\sigma_m^{F_y}(x)$ and $K_m^{F_y}$ can be written as follows

$$\sigma_m^{F_y}(x)=-\frac{6M'_{F_y}}{bS^2}\left(1-\frac{2x}{S}\right)=-\frac{6F_y\left(x-\frac{S}{2}\right)}{bS^2}\left(1-\frac{2x}{S}\right) \tag{20}$$

$$K_m^{F_y}=\frac{6FL}{bS^2}\left(\frac{\sin\phi S}{2L}\right)\sqrt{\pi a}\times\frac{\sqrt{2}}{\pi}\int_0^a\left(1-\frac{2x}{S}\right)^2\times\left[1+m_1\left(\frac{a-x}{a}\right)+m_2\left(\frac{a-x}{a}\right)^2\right]\frac{dx}{\sqrt{a(a-x)}} \tag{21}$$

The stress and stress intensity factor become

$$\sigma(x)=\sigma_m(x)-\sigma_{y0}-\sigma_m^{F_y}(x) \tag{22}$$

$$K=K_m+K_{F_y}+K_m^{F_y} \tag{23}$$

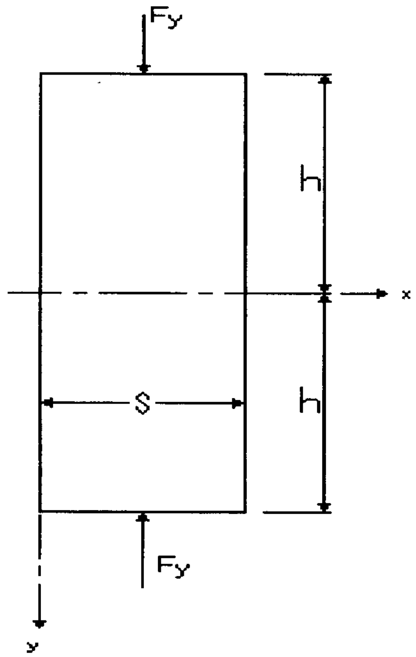

Figure 3 Model of the second approach.

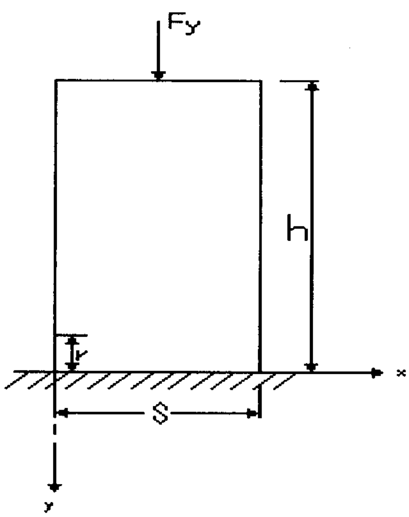

Figure 4 Model of the third approach.

B. Second Approach

Model of a rectangular plate in compression is shown in Figure 3. Timoshenko[6] derived the stress $\sigma_m^{F_y}(x)$ at y=0 by theory of elasticity as a summation form.

$$\sigma_m^{F_y}(x)=\frac{2F_y}{bS}\sum_{m=1}^{\infty}\left[\left(\frac{m\pi h}{S/2}+1\right)e^{-\frac{m\pi h}{S/2}}\times\cos\left(\frac{m\pi\left(x-S/2\right)}{S/2}\right)\right] \tag{24}$$

The stress intensity factor corresponding to equation (24) is

$$K_m^{F_y}=-\frac{2F_y}{bS}\sqrt{\pi a}\times\frac{\sqrt{2}}{\pi}\int_0^a\sum_{m=1}^{\infty}\left[\left(\frac{m\pi h}{S/2}+1\right)\times e^{-\frac{mh}{S/2}}\cos\left(\frac{m\pi\left(x-S/2\right)}{S/2}\right)\right]\left[1+m_1\left(\frac{a-x}{a}\right)+m_2\left(\frac{a-x}{a}\right)^2\right]\frac{dx}{\sqrt{a(a-x)}} \tag{25}$$

The mode I stress intensity factor of cracked gear tooth can be obtained by substituting equation (25) into equation (23).

C. Third Approach

A modified model for simulating cracked gear tooth is shown in Figure 4 in which one end is fixed. For this model, the boundary is more closed to a real gear than the other two models. In order to decrease the concentration of stress $\sigma_m^{F_y}(x)$, the position of fillet radius y=r is considered.

$$\sigma_m^{F_y}(x)=\frac{2F_y}{bS}\sum_{m=1}^{\infty}\left[\left(\frac{m\pi}{S/2}\right)^2\cos\left(\frac{m\pi\left(x-S/2\right)}{S/2}\right)f(r)\right] \tag{26}$$

where

$$f(r)=C_1\cosh\frac{m\pi r}{S/2}+C_2\sinh\frac{m\pi r}{S/2}+C_3 r\cosh\frac{m\pi r}{S/2}+C_4 r\sinh\frac{m\pi r}{S/2}$$

$$C_1=\frac{1/\alpha^2}{C_8+C_7\times\dfrac{\left[\dfrac{m\pi(1+\nu)}{S}C_5-\dfrac{m\pi}{S/2}\sinh\dfrac{m\pi h}{S/2}\right]}{\left[\dfrac{m\pi(1+\nu)}{\frac{S}{2}(1-\nu)}C_6+\dfrac{m\pi}{S/2}\cosh\dfrac{m\pi h}{S/2}\right]}}$$

$$C_2 = \frac{\left[\frac{m\pi(1+\nu)}{S}C_5 - \frac{m\pi}{S/2}\sinh\frac{m\pi h}{S/2}\right]}{\left[\frac{m\pi(1+\nu)}{\frac{S}{2}(1-\nu)}C_6 + \frac{m\pi}{S/2}\cosh\frac{m\pi h}{S/2}\right]}C_1$$

$$C_3 = \frac{m\pi(1+\nu)}{\frac{S}{2}(1-\nu)}C_2$$

$$C_4 = -\frac{m\pi(1+\nu)}{S}C_1$$

$$C_5 = \sinh\frac{m\pi h}{S/2} + \frac{m\pi h}{S/2}\cosh\frac{m\pi h}{S/2}$$

$$C_6 = \cosh\frac{m\pi h}{S/2} + \frac{m\pi h}{S/2}\sinh\frac{m\pi h}{S/2}$$

$$C_7 = \sinh\frac{m\pi h}{S/2} + \frac{m\pi h}{S/2}\frac{(1+\nu)}{(1-\nu)}\cosh\frac{m\pi h}{S/2}$$

$$C_8 = \cosh\frac{m\pi h}{S/2} - \frac{m\pi(1+\nu)}{S}h\sinh\frac{m\pi h}{S/2}$$

Therefore, $K_m^{F_y}$ can be derived as

$$K_m^{F_y} = -\frac{2F_y}{bS}\sqrt{\pi a}\times\frac{\sqrt{2}}{\pi}\int_0^a\sum_{m=1}^{\infty}\left[\cos\left(\frac{m\pi\left(x-\frac{S}{2}\right)}{\frac{S}{2}}\right)\right.$$
$$\left.\times f(r)\right]\left[1+m_1\left(\frac{a-x}{a}\right)+m_2\left(\frac{a-x}{a}\right)^2\right]\frac{dx}{\sqrt{a(a-x)}} \quad (27)$$

IV. Results and Discussion

The stress intensity factor of cracked gear tooth can not be determined directly by fatigue experiment although it is the main parameter of effect of fatigue crack growth. Therefore, many numerical methods for this subject had been presented. A well-known numerical method, weight function method, is used to determine the stress intensity factor of cracked gear tooth in this study. In order to make sure that is a correct and believable result, the result of finite element method for mode I stress intensity factor of cracked gear tooth by Lewicki[2] is referred.

In this study the tooth thickness of the position of crack S=5.81mm, applied force F=2700N, working pressure angle $\phi = 20°$, fillet radius of tooth root r=1.5mm, whole depth of gear tooth h=7.62mm, face width of gear tooth b=7.62mm. The distances L=3.4mm and c=2.29mm are chosen. Figure 5 contains five results that are all used to describe the mode I stress intensity factor of cracked gear tooth. The same conditions of cracked gear tooth with Lewicki[2] are considered. The results of weight function method by Flasker[5] and finite element method have a similar tendency of the change of mode I stress intensity factor

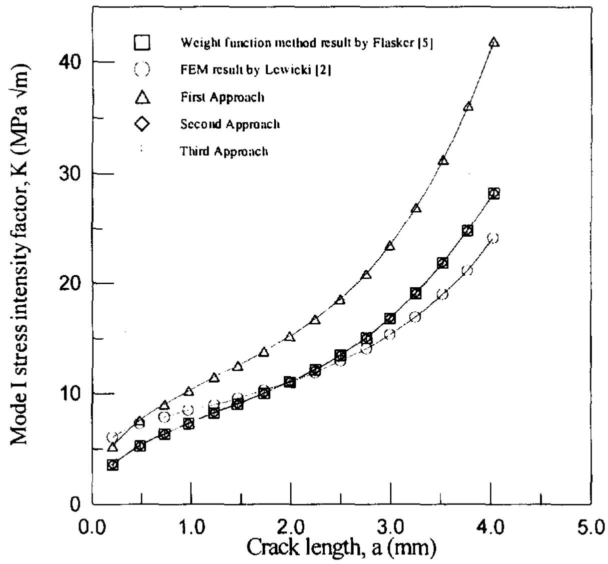

Figure 5 Mode I stress intensity factor of cracked gear tooth.

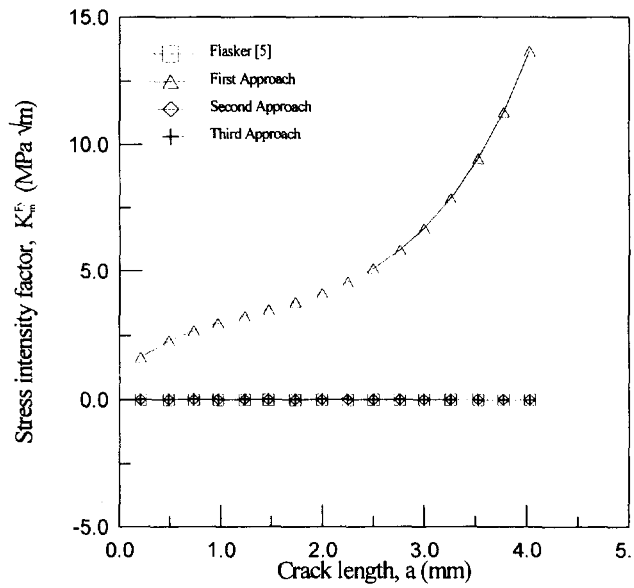

Figure 6 Stress intensity factor due to point force F_y.

depending on the crack length. In addition, three curves of modified models are obtained by adding the stress $\sigma_m^{F_y}(x)$ in the stress field presented by Flasker[5].

Figure 6 shows the comparison of $K_m^{F_y}$ due to point force F_y by Flasker[5], first approach, second approach and third approach. In fact, the moment M'_{F_y} should be a function of position, and is defined as a linear distribution by the first approach based on theory of solid mechanics, but are defined as a cosine distribution by the second and third approaches based on theory of elasticity. It becomes the major difference that the stress intensity factor of first approach is much higher than others.

The curves of result by Flasker[5], second approach and third approach seem to overlap in Figure 5, but the differences can be observed from Figure 7 . It indicates that $K_m^{F_y}$ due to end effect considered in third approach at gear tooth root is much greater than the symmetrical rectangular plate of second approach and Flasker [5]. Compare to the summation of K in equation (23), the $K_m^{F_y}$ of cracked gear tooth is very small. Therefore, the total K used model of third approach in this study does not have the obvious difference with the result by Flasker[5] shown in Figure 5.

Equations (3) and (4) are presented by Flasker[5] to determine the coefficients m_1 and m_2 in weight function for this problem. Because both of the two coefficients are functions of crack length, the repeated processes of calculation of integration and differentiation for each fatigue cycle are quite complicated.

Therefore, these two equations can be simplified by curve fitting as follows:

$$m_1 = 0.6428 + 0.3665\alpha + 25.0033\alpha^2 - 49.1034\alpha^3 + 67.7017\alpha^4 \tag{28}$$

$$m_2 = 0.1572 - 1.8931\alpha + 16.9193\alpha^2 - 38.7784\alpha^3 + 55.9269\alpha^4 \tag{29}$$

The good agreement for comparison of the coefficients m_1 and m_2 by Flasker[5] and curve fitting are shown in Figure 8 and Figure 9, respectively. Equation (28) and equation (29) hence make the determination of the coefficients m_1 and m_2 more efficient and convenient.

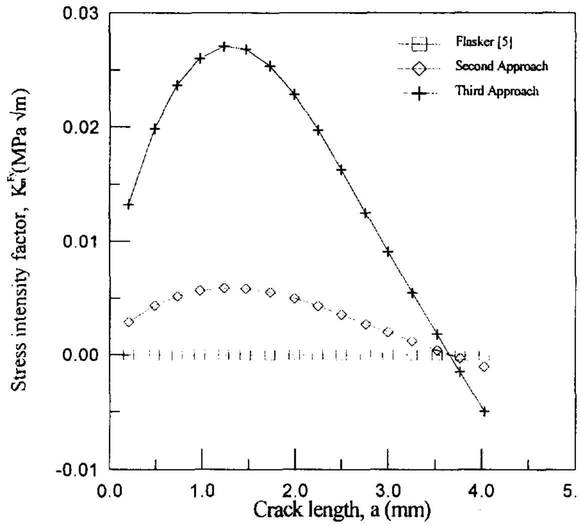

Figure 7 Comparison of stress intensity factors due to point force F_y for several approaches

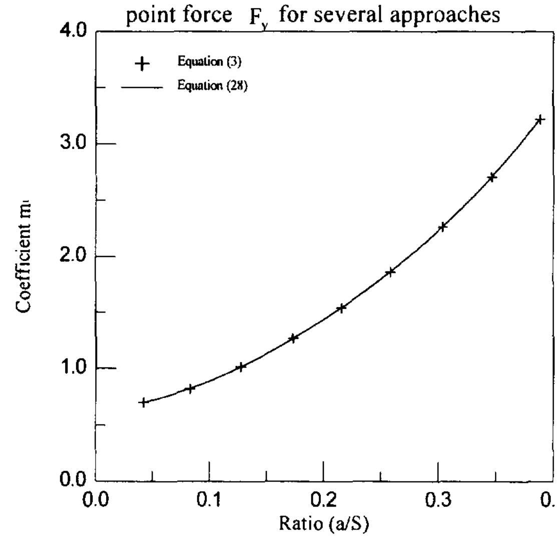

Figure 8 Coefficient m_1 versus ratio (a/s).

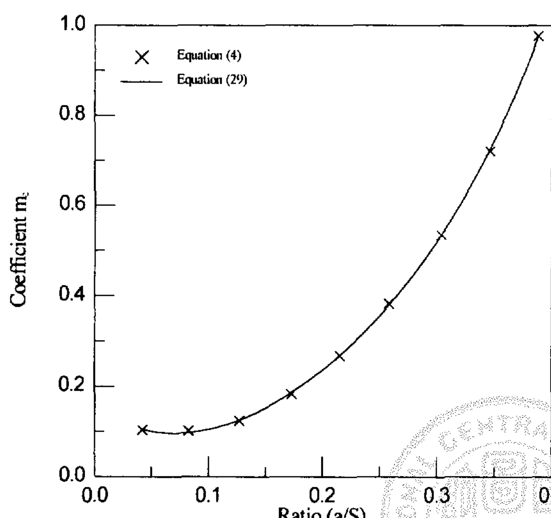

Figure 9 Coefficient m_2 versus ratio (a/s).

V. Conclusion

In this study the stress intensity factor of a cracked gear tooth is determined by the weight function method. The accuracy of results is dependent on the stress and the selected weight function. The weight function of a cracked gear tooth has been presented by Flasker[5]. The stress fields due to F_x and M_{F_y} have been considered in Flasker's work[5], but the bending stress due to the point force F_y in Figure 2 is considered as the constant through the cross section by Flasker[5]. In this study the point force has been considered by three approaches. In the first approach the bending moment M'_{F_y} due to point force is considered. Timoshenko's solution of stress for the plate shown in Figure 3 is used in the second approach. But the boundary conditions in second approach are not equivalent to the gear tooth. In reality, the gear tooth can be simulated in Figure 4 and the stress field is solved in the third approach. The stress intensity factors by three approaches and Flasker's approach have been compared in Figure 6. The result of the first approach is different to others. In Figure 7, second, third and Flasker's approaches are compared. Since the third approach is closed to a real gear tooth, the third approach is a best one in this study. But the total stress intensity factors have no difference among second, third and Flasker's approaches shown in Figure 5.

In addition, the weight functions shown in equations (3) and (4) are determined by a complicated integration. To simplify the calculation, two polynomial equations are presented as equations (28) and (29) by curve fitting. The comparisons between polynomial equations and integrated equations have been shown in Figures 8 and 9, that the good agreement is obtained. These polynomial equations of weight functions provide the easy and convenient determination of stress intensity factor.

References

1. S. Pehan, T. K. Hellen and J. Flasker, "Applying numerical methods for determining the service life of gears", Fatigue & Fracture of Engineering Materials & Structures., Vol. 18, No. 9, pp. 971-979, 1995.
2. D. G. Lewicki and R. Ballarini, "Effect of rim thickness on gear crack propagation path", Journal of Mechanical Design., Vol. 119, pp. 88-95, 1997.
3. J. Flasker, S. Glodez and S. Pehan, "Influence of contact area on service life of gears with crack in tooth root", Communications in Numerical Methods in Engineering., Vol. 11, pp. 49-58, 1995.
4. S. R. Daniewicz, J. A. Collins and D. R. Houser, "The stress intensity factor and stiffness for a cracked spur gear tooth", Journal of Mechanical Design., Vol. 116, pp. 697-700, 1994.
5. B. Abersek and J. Flasker, "Stress intensity factor for cracked gear tooth", Theoretical and Applied Fracture Mechanics., Vol. 20, pp. 99-104, 1994.
6. [S. P. Timoshenko and J. N. Goodier, Theory of elasticity, 3rd edition, McGraw-Hill, Inc., pp. 35-63, 1970.

以權函數法對含裂紋齒輪應力強度因子之研究

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摘　要

對於齒輪含齒根裂紋的齒之應力強度因子，利用適當的應力場以及權函數的積分計算，可求得較為正確之結果。在本文中，裂齒所受之負載可分解為一在 x 方向的點力，一在 y 方向之點力，以及一彎曲力矩作用。在 Flasker 的假設當中，由 y 方向點力所造成的應力場被視為常數；在本文中，對於此彎矩所造成之應力場可以三種方式來探討。在第三種方式中，是以一端固定之平板來模擬裂齒之狀態，並以彈性力學的理論來求解裂齒所受到之應力場；因此可得到較為合理之裂齒所受應力場以及裂紋尖端之應力強度因子。此外，Flasker 在求取權函數時所用的方程式，皆須經過多次反覆的微分與積分計算，過程太過於繁雜，因此，本文所提出的多項式可更容易且簡便的用以計算權函數。

關鍵詞：權函數法，應力強度因子，齒輪。