Chung Yuan Journal
Vol. 27. No. 1, 1999. pp. 1-6

MUSCL Type Algorithm for Acoustic Wave Equations in Heterogeneous Media

KAI-HUI CHEN, CHIH-WEI LIAO AND TIEN-YU SUN

Department of Mathematics
Chung Yuan Christian University
Chung-Li, Taiwan 32023, R.O.C.

(Received: July 30, 1998; Accepted: August 31, 1998)

ABSTRACT

A MUSCL type algorithm for solving the acoustic wave equations numerically in a heterogeneous medium is developed. Our numerical tests show that, as compared to LeVeque's wave propagation algorithm given in [1], the present MUSCL type algorithm is more accurate and is more stable for large time.

Key words: *Acoustic wave equations, hyperbolic conservation laws, MUSCL type algorithm, Riemann solver, wave propagation algorithms.*

I. INTRODUCTION

In recent years, the investigation on high resolution methods for solving hyperbolic conservation laws numerically has been quite successful. See [2] and its references for the recent development in this field. What should be done next is to generalize these methods to solve more general hyperbolic systems. In real applications, a lot of problems are of the type

$$q_t + A(x)q_x = 0 \tag{1.1}$$

where $q(x, t) \in \mathbf{R}^n$, $A(x) \in \mathbf{R}^{n\times n}$. The coefficient matrix $A(x)$ is allowed to have discontinuities at some locations in the domain considered. At each discontinuity of $A(x)$, jump condition

$$[q] = 0 \tag{1.2}$$

is considered along with (1.1). Here $[q]$ represents the jump in q across the interface of discontinuity. Typical examples are the propagation of acoustic waves and elastic waves in heterogeneous media.

From what has been found in the theory of numerical hyperbolic conservation laws, even for linear hyperbolic systems, nonlinear methods have to be used in order to achieve second order accuracy and be monotone preserving. See Godunov Theorem in Chapter 16 of [2]. In solving (1.1) and (1.2), further complication is caused by finding the correct way to incorporate the jump condition (1.2) into the numerical method. Recently, in [1], LeVeque showed that we can incorporate the jump condition into the numerical method through the use of a proper Riemann solver. The model problem considered in [1] is the propagation of acoustic waves in heterogeneous media. The governing equations are

$$p_t + k(x)u_x = 0,$$

$$\rho(x)u_t + p_x = 0 \tag{1.3}$$

Here the unknowns are the pressure disturbance p and the velocity u. The coefficient functions $\rho(x)$ and $k(x)$ are the density and the bulk modulus of elasticity. In case $\rho(x)$ and $k(x)$ are discontinuous, on each interface of discontinuity, jump conditions

$$[u] = 0, \quad [p] = 0 \tag{1.4}$$

are added to the system (1.3). (1.4) requires that the pressure disturbance p and the velocity field u to be continuous across interfaces between different media. See [3] for the derivation of the jump conditions.

In § 2, we rewrite the acoustic waves equations in conservation form and develop a MUSCL type algorithm using the Riemann solver purposed by LeVeque in [1]. The resulting algorithm is then tested thoroughly in § 3 and is compared to the wave propagation algorithm given developed by LeVeque in [1]. A detail comparison of our MUSCL type algorithm and LeVeque's wave propagation algorithm is then given in § 4.

II. MUSCL TYPE ALGORITHM

Note that the one dimensional acoustic wave equations in (1.3) is of the form (1.1) in which $q(x, t)$ is the transpose of $(p(x, t), u(x, t))$;

$$A(x) = \begin{bmatrix} 0 & k(x) \\ 1/\rho(x) & 0 \end{bmatrix}. \tag{2.1}$$

To derive a MUSCL type algorithm for the one dimensional acoustic wave equations, we first rewrite them as a conservation law of the form

$$\psi_t + f(\psi)_x = 0, \tag{2.2}$$

where $\psi = A^{-1}(x)q$ and $f(\psi) = q$.

For a hyperbolic conservation law of the form

$$w_t + f(w)_x = 0, \tag{2.3}$$

the corresponding MUSCL scheme is

$$w_j^{n+1} = w_j^n - \frac{k}{h}(g_{j+1/2}^n - g_{j-1/2}^n) \tag{2.4}$$

where

$$g_{j-1/2}^n = f(w_R(0; w_{j-1/2,-}^{n+1/2}, w_{j-1/2,+}^{n+1/2})).$$

In the evaluation of numerical flux $g_{j-1/2}^n$, the function w_R is the solution of the Riemann problem of (2.3) with Riemann data $w_{j-1/2,\pm}^{n+1/2}$;

$$w_{j-1/2,-}^{n+1/2} = w_{j-1/2,-}^n - \frac{k}{2h}(f(w_{j-1/2,-}^n) - f(w_{j-3/2,+}^n)), \tag{2.5}$$

$$w_{j-1/2,+}^{n+1/2} = w_{j-1/2,+}^n - \frac{k}{2h}(f(w_{j+1/2,-}^n) - f(w_{j-1/2,+}^n)), \tag{2.6}$$

where

$$w_{j-1/2,-}^n = w_{j-1}^n + \frac{S_{j-1}^n}{2},$$

$$w_{j-1/2,+}^n = w_j^n - \frac{S_j^n}{2}. \tag{2.7}$$

Here S_j^n/h is used to approximate w_x at the point (x_j, t_n). For example, we can set S_j^n to be the vector whose components consist of applying the minmod function minmod(·, ·) on the components of $w_{j+1}^n - w_j^n$ and $w_j^n - w_{j-1}^n$. For real numbers a and b, the minmod function minmod(·, ·) is defined by

$$\text{minmod}(a, b) = \frac{1}{2}(\text{sgn}(a) + \text{sgn}(b)) \min(|a|, |b|).$$

See [4] and [5] for the details of MUSCL scheme.

To apply the MUSCL scheme above to the one dimensional acoustic wave equations in the form (2.2), we need to introduce the Riemann solver given by LeVeque in [1]. Let ρ_j, k_j be the material parameters for the j^{th} cell and let $A_j = A(x_j)$ where $A(x)$ is the coefficient matrix given in (2.1). Note that A_j has eigenvalues and eigenvectors

$$\lambda_j^1 = -c_j, \quad \lambda_j^2 = c_j,$$

$$\gamma_j^1 = \begin{bmatrix} -c_j \\ 1/\rho_j \end{bmatrix}, \quad \gamma_j^2 = \begin{bmatrix} -c_j \\ 1/\rho_j \end{bmatrix} \tag{2.8}$$

Consider the Riemann problem of (1.3) with Riemann data q_{j-1}^n and q_j^n. The left-going wave moves into the $(j-1)^{st}$ cell with velocity $s_j^1 = -c_{j-1}$. The jump across this wave must be a scalar multiple of the eigenvector γ_{j-1}^1 of the matrix A_{j-1}. Let

$$\omega_j^1 = \alpha_j^1 \gamma_{j-1}^1. \tag{2.9}$$

The right-going wave moves into the j^{th} cell with velocity $s_j^2 = c_j$. The jump is a scalar multiple of the eigenvector γ_j^2 of matrix A_j. Let

$$\omega_j^2 = \alpha_j^2 \gamma_j^2. \tag{2.10}$$

On the cell interface $x = x_{j-1/2}$, the jump conditions (1.4) require that the primitive variables p, u be continuous across the interface. That is to say,

$$q_{j-1}^n + \omega_j^1 = q_{j-1/2}^n = q_j^n - \omega_j^2. \tag{2.11}$$

By solving the algebraic linear system resulting from(2.11), we can determine α_j^1 and α_j^2.

Note that the one dimensional acoustic wave equations, in conservation form (2.2), have $\psi_j^n = A_j^{-1} q_j^n$. First, apply (2.5)-(2.7) with ψ_j^n in place of ω_j^n. Let the resulting intermediate states be $\psi_{j-1/2,\pm}^{n+1/2}$. Next, we transform the intermediate states $\psi_{j-1/2,\pm}^{n+1/2}$ from the conservative formulation to the primitive formulation. Set

$$\bar{q}_{j-1}^n = A_{j-1}\psi_{j-1/2,-}^{n+1/2},$$

$$\bar{q}_j^n = A_j \psi_{j-1/2,+}^{n+1/2}. \tag{2.12}$$

Now apply LeVeque's Riemann solver given above with $\bar{q}_{j-1}^n$ and $\bar{q}_j^n$ as the Riemann data to obtain the resulting state $\bar{q}_{j-1/2}^n$ on the interface $x = x_{j-1/2}$; i.e.,

$$\bar{q}_{j-1}^n + \omega_j^1 = \bar{q}_{j-1/2}^n = \bar{q}_j^n - \omega_j^2.$$

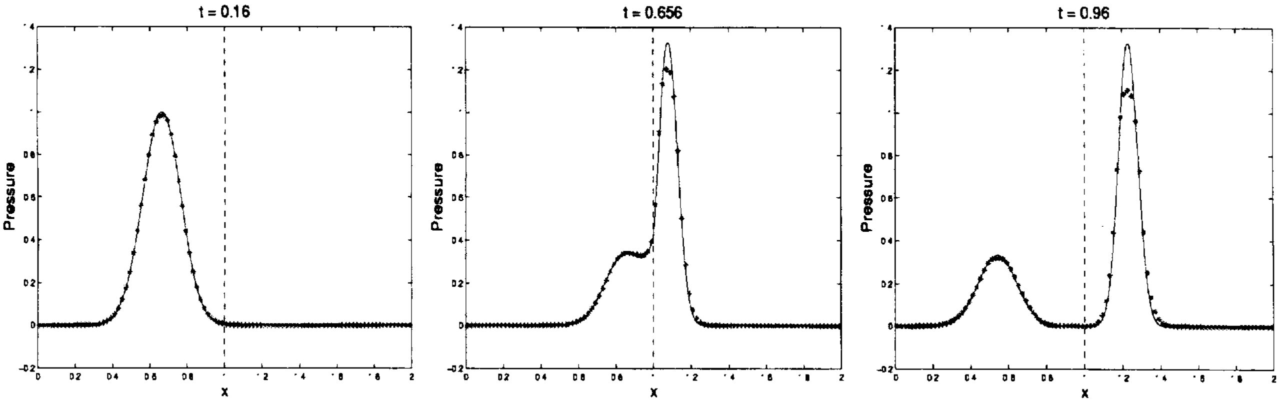

Fig. 1. Profiles of p by (2.13) at $t = 0.16$, 0.656 and 0.96. The solid lines are the graphs of the fine grid solution.

We can now set the numerical flux $f(\psi_R(0; \psi^{n+1/2}_{j-1/2,-}, \psi^{n+1/2}_{j-1/2,+}))$ equal to $\bar{q}^n_{j-1/2}$ where $\psi_R(\bullet; \psi^{n+1/2}_{j-1/2,-}, \psi^{n+1/2}_{j-1/2,+})$ is the solution of the Riemann problem of (2.2) with Riemann data $\psi^{n+1/2}_{j-1/2,\pm}$. Now the MUSCL type algorithm for (2.2) takes the form

$$\psi^{n+1}_j = \psi^n_j - \frac{k}{h}(\bar{q}^n_{j+1/2} - \bar{q}^n_{j-1/2}). \tag{2.13}$$

III. NUMERICAL RESULTS

To begin with, we consider the case in which the heterogeneous medium has only one interface separating two different material. A simple account of an acoustic wave hitting an interface separating two media with constant physical parameters ρ, k is given in Chapter 3 of [6]. When the impedance $Z = \rho c$ of each medium is different, part of the acoustic wave is transmitted across the interface and part of it is reflected. In case Z is constant across the interface, the acoustic wave is completely transmitted across the interface and nothing is reflected. See [6] for the details. In the following test problems, we will compare the numerical results generated by the MUSCL type algorithm (2.13) in § 2 to those resulting from the wave propagation algorithm given in [1] by LeVeque. In all the test problems below, the minmod function is used as the limiter in both algorithms and the zero extrapolation boundary condition is used at both endpoints of the computational domain.

Example 1. Suppose that the medium considered consists of two different material separated by an interface located at $x = 1$, in which, $k(x) = 1$ for all x and

$$\rho(x) = \begin{cases} 1, & \text{if } x < 1 \\ 4, & \text{if } x > 1 \end{cases}.$$

The initial conditions chosen are

$$p(x, 0) = u(x, 0) = e^{-48(x-0.5)^2}.$$

Apply (2.13) with $h = 0.02$, $k = 0.016$ and have the result compared to the fine grid numerical solution (p_f, u_f) computed by (2.13), using $h = 0.00125$ and $k = 0.001$.

Figure 1 indicates that (2.13) is capable of providing fairly good resolution of the transmitted and reflected waves. Although (2.13) is a nonlinear method for solving (1.3), the linear behavior of (1.3) is well captured. The transmitted wave and the reflected wave head toward different directions without any noticeable interference with each other. Note that the sound speed c is equal to 1 to the left of $x = 1$ and is equal to 0.5 to the right of $x = 1$. According to the theory in [6], the transmitted wave should be only half as wide as the incident wave. Our result in Figure 1 clearly indicates so. However the same result also points out a typical phenomenon in the computation of propagation of acoustic wave over heterogeneous media. Even the grid system is fine enough to resolve the incident wave, it may not be fine enough for the transmitted wave. In this respect, higher order methods may be valuable in such computations. Replacing (2.13) with LeVeque's wave propagation algorithm will give results very close to the ones given above. See [1] for similar tests.

Next we compare the order of accuracy of LeVeque's wave propagation algorithm and (2.13) by repeating the same computation using mesh sizes $h = 0.04/2^\ell$, $\ell = 1, 2, 3, 4$ and time steps $k = 0.8h$. At $t = 0.16$, 0.656 and 0.96, we plot $\log(\|E^n\|_1)$ against $\log h$ where $E^n = p^n - p_f$. See Figure 2 below.

The orders of accuracy of the two methods in 1-norm at these three instants are given in Table 1. Before the incident wave hits the interface of discontinuity, the order of accuracy of LeVeque's wave propagation algorithm and the MUSCL type algorithm developed in § 2 are about the same. Once the transmitted wave and the reflected wave are generated, the MUSCL type algorithm is significantly

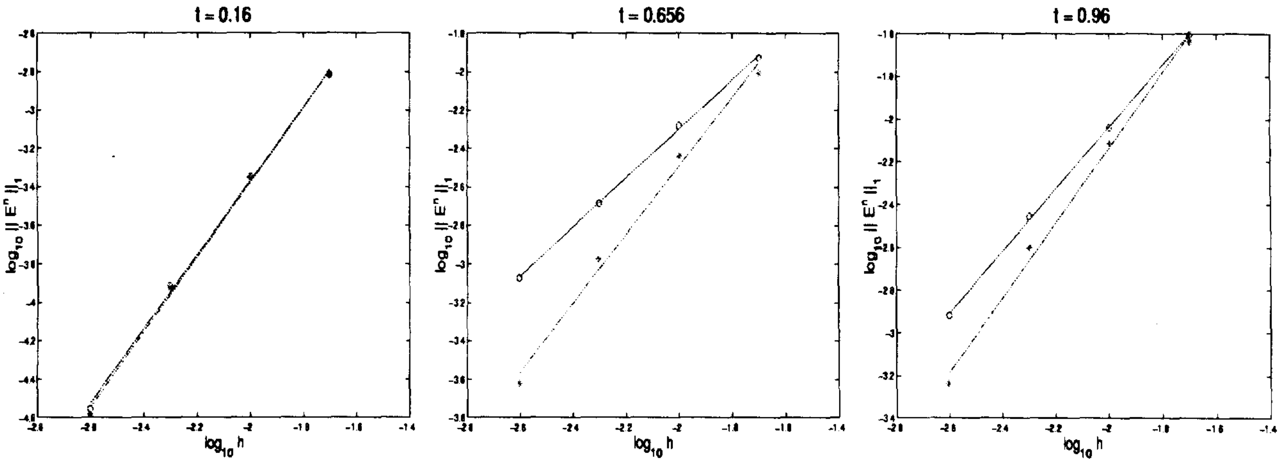

Fig. 2. Error analysis in 1-norm. o : LeVeque's wave propagation algorithm; * : MUSCL type algorithm (2.13).

Table 1. Order of accuracy in 1-norm.

method	$t = 0.16$	$t = 0.656$	$t = 0.96$
LeVeque's	1.92	1.28	1.45
MUSCL	1.94	1.79	1.75

more accurate than LeVeque's wave propagation algorithm.

Now we consider the case in which the heterogeneous medium has a large number of recurrent cells; each cell is composed of two different material separated by an interface.

Example 2. For $\epsilon > 0$ and $-1 \le x \le 199$, let $\rho(x) = \rho_*(x/\epsilon)$ and $k(x) = k_*(x/\epsilon)$, where

$$\rho_*(\xi) = \begin{cases} 1, & \text{if } \xi < 0.5 \\ \rho_R, & \text{if } \xi > 0.5 \end{cases},$$

$$k_*(\xi) = \begin{cases} 1, & \text{if } \xi < 0.5 \\ k_R, & \text{if } \xi > 0.5 \end{cases}. \qquad (3.1)$$

Here ρ_R, k_R can be any positive constant. The initial conditions chosen are

$$p(x, 0) = 0,$$

$$u(x, 0) = e^{-\alpha(x-1)^2}. \qquad (3.2)$$

First we set $\rho_R = k_R = 4$, $\alpha = 3.454$ and $\epsilon = 0.4$. Choose $h = 0.01$, $k = 0.008$ and run LeVeque's wave propagation algorithm up to $t = 160$. The time history of the maximal and minimal values of p at each instant and the profile of p at $t = 160$ are plotted in Figure 3.

Similar testings can be found in [7] for $\rho_R = k_R = 3$ and various α. These testings in [7] indicate that the solutions of the acoustic wave equations possess a dispersive feature. Furthermore, the amplitudes of the wave fronts are

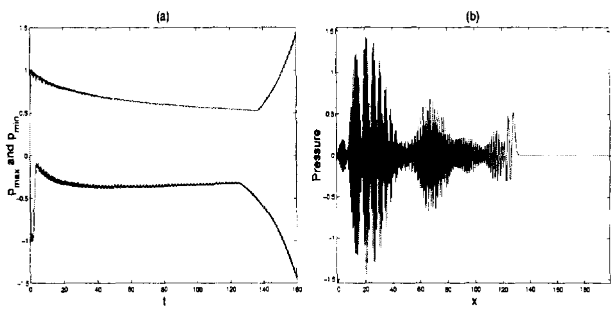

Fig. 3. Method : LeVeque's wave propagation algorithm. (a) : Maximal and minimal values of p. (b) : p at $t = 160$.

considerably larger than those behind them. However, in Figure 3, we see that the numerical solution of LeVeque's wave propagation algorithm has a completely different profile. And the range for p at each instant seems to be increasing. If we maintain $\rho_R = k_R$ and keep increasing their values, or if we use a smaller $\epsilon > 0$, then more severe instability of LeVeque's wave propagation algorithm will occur. Also see [8] for other numerical tests on instability of LeVeque's wave propagation algorithm.

Now do the same computation again using (2.13). The resulting plots in Figure 4 resemble those in [7].

Finally we set $\rho_R = 4$, $k_R = 1/4$ with α and ϵ remain the same. Note that now the impedance $Z = 1$ throughout the whole region. From Figure 5 below, we see that LeVeque's wave propagation algorithm remains stable, and there is no wave packet at all behind the wave front. That is to say, there is no reflected wave generated by any of the interfaces of discontinuity. Replacing LeVeque's wave propagation algorithm with the MUSCL type algorithm will lead to a similar result.

From Figure 3 and Figure 5, we suspect that the insta-

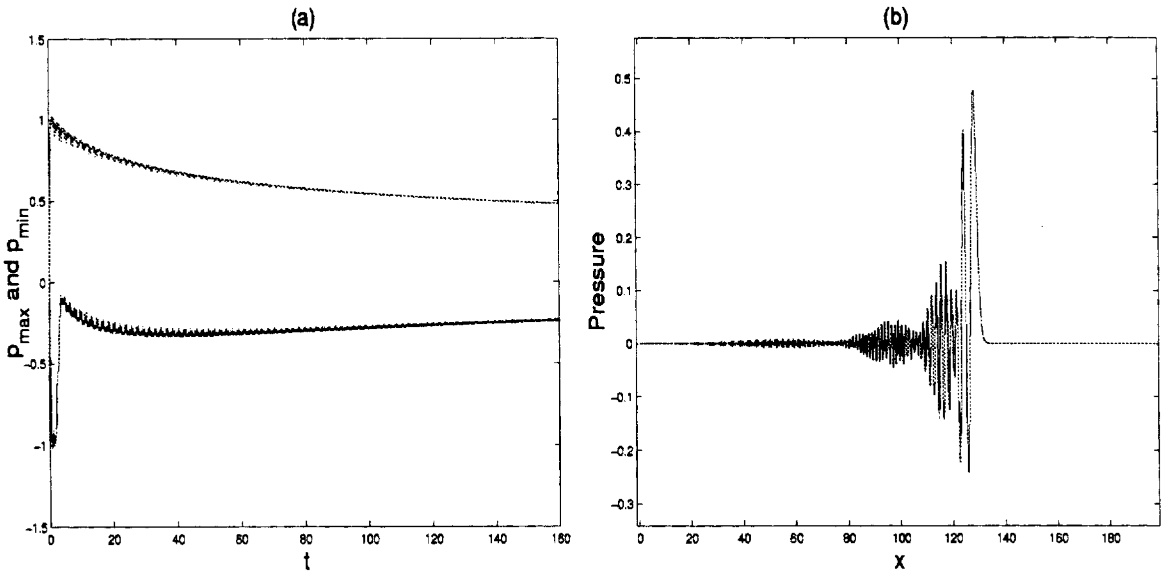

Fig. 4. Method : MUSCL type algorithm (2.13). (a) : Maximal and minimal values of p. (b) : p at $t = 160$.

Impedance $Z = 1$.

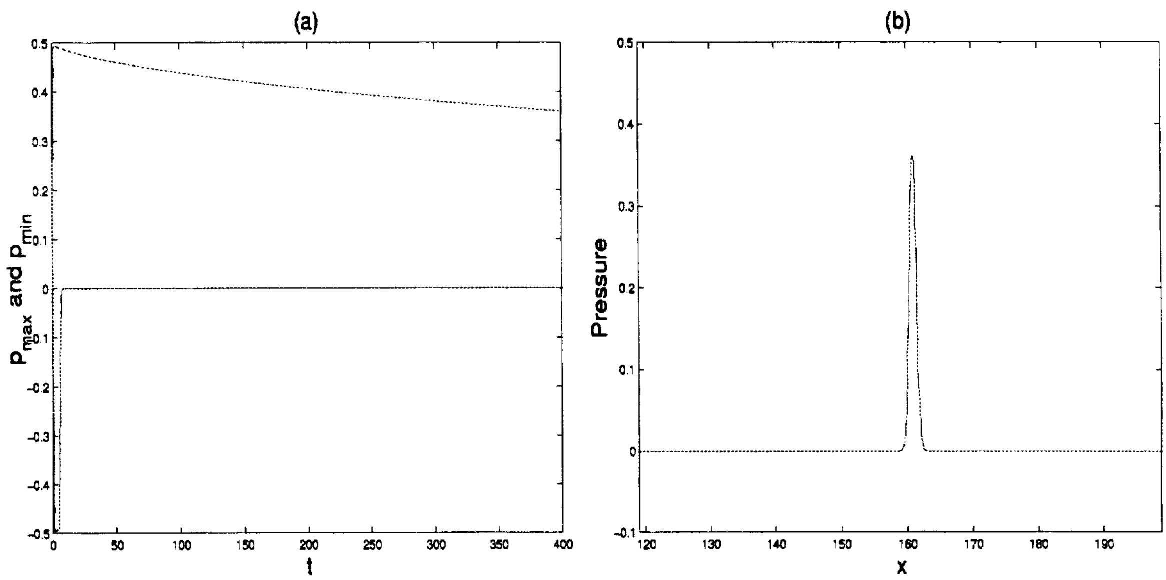

Fig. 5. Method : LeVeque's wave propagation algorithm. (a) : Maximal and minimal values of p. (b) : p at $t = 160$.

bility of LeVeque's wave propagation algorithm for large time in problem with varying impedance Z may be due to the nonlinear resonance of the wave train behind the wave front and the periodic cell structure of the heterogeneous medium.

IV. CONCLUSIONS

A MUSCL type algorithm (2.13) for solving the one dimensional acoustic wave equations in heterogeneous media is given in § 2. In contrast to LeVeque's approach in [1], the derivation of the present algorithm relies heavily on rewriting the one dimensional acoustic wave equations (1.3) in conservation form (2.2). Then the MUSCL scheme for hyperbolic conservation laws is applied together with LeVeque's Riemann solver for one dimensional acoustic wave equations, which enables the jump conditions (1.4) to be incorporated into the numerical method. As our numerical results in § 3 show, the resulting MUSCL type algorithm (2.13) is more accurate and has much better large time stability as compared to LeVeque's wave propagation algorithm. This indicates that, when we extend the theory of numerical conservation laws to more general hyperbolic systems, it is better to try to rewrite the hyperbolic system in conservation form. In the future, we will extend the work in this paper to develop finite volume type flux differencing method for numerical computation of two dimensional acoustic waves and elastic waves in heterogeneous medium.

REFERENCES

1. R. J. LeVeque, *Wave Propagation Algorithm for Multidimensional Hyperbolic Systems*, J. Comput. Phys., Vol. 131, pp. 327-353 (1997).
2. R. J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhäuser, Basel (1990).
3. Chaoming Zhang, *Immersed Interface Methods for Hyperbolic Systems of Partial Differential Equations with Discontinuous Coefficients*, PhD Thesis, University of Washington (1996).
4. Phillip Colella and Harland M. Glaz, *Efficient Solution Algorithms for the Riemann Problem for Real Gases*, J. Comput. Phys., Vol. 59, pp. 264-289 (1985).
5. Edwige Godlewski and Pierre-Arnaud Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Springer-Verlag, New York, U.S.A. (1996).
6. N. Bakhvalov and G. Panasenko, *Homogenisation: Averaging Processes in Periodic Media*, Kluwer Academic Publishers, London (1989).
7. F. Santosa and W. W. Symes, *A Dispersive Effective Medium for Wave Propagation in Periodic Composities*, SIAM J. Appl. Math., Vol. 51, pp. 984-1005 (1991).
8. T. Fogarty, *High-Resolution Finite Volume Methods for Acoustics in a Rapidly-Varying Heterogeneous Medium*, University of Washington, Master Thesis (1997).

複合介質上聲波方程組的MUSCL型演算法

陳開煇　廖志偉　孫天佑

中原大學數學研究所

臺灣省中壢市普仁22號

摘　要

在本文中，我們針對複合介質上聲波方程組的數值計算，發展MUSCL型演算法。從第三節的數值實驗中，我們發現和LeVeque在[1]中所提的波傳遞法相比較，上述MUSCL型演算法具有較高的精確度及較佳的長時間穩定性。[1]和本文中的結果顯示，在把雙曲線型守恆律的數值方法應用到更一般的雙曲線型偏微方程時，應嘗試把問題轉換成雙曲線型守恆律爲宜。同時LeVeque在[1]中所提，經由Riemann solver把不同介質間的不連續條件引入數值方法中爲一有效可行之法。

關鍵詞：聲波方程組，雙曲線型守律，MUSCL型演算法，Riemann solver，波傳遞法。