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Generalized Weber Equations via Fractional Calculus

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ABSTRACT

Based on Nishimoto's fractional calculus [1], the particular solutions to the well-known special second order differential equations, such as Gauss, Legendre, Jacobi, Tchebycheff, and Coulomb have been obtained. As for their generalized form, their particular solutions are discussed via fractional calculus method by some authors ([2]~[7]). Recently, in 1999, S.T.Tu, et al. [8] have treated the particular solution to the generalized Nth order equations, such as Associated Legendre, Euler, and Hermite equations.

In 1998, K. Nishimoto [10], obtained a particular solution to the famous second order Weber equation which appeared in quantum mechanics by using his N^{ν} method.

In this paper, the solution to its generalized Weber equation by using fractional calculus method will be discussed in detail with some examples given.

Key words: *Fractional calculus, Generalized Leibniz's Rule, Generalized Weber's equation.*

I. Introduction

A. Definition

Let D = {D_-, D_+}, C = {C_-, C_+}

C_- be a curve along the cut joining two points z and $-\infty + i\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-, D_+ be a domain surrounded by C_+.

(Here D_+ contains the points over the curve C.)

Moreover, let $f = f(z)$ be a regular function in D ($z \in D$),

$$f_\upsilon = (f)_\upsilon = {}_C(f)_\upsilon = \frac{\Gamma(\upsilon+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\upsilon+1}} d\zeta \, (\upsilon \notin \mathbf{Z}^-)$$

$$(f)_{-m} = \lim_{\upsilon \to -m} (f)_\upsilon \, (m \in \mathbf{Z}^+),$$

where $-\pi \leq \arg(\zeta - z) \leq \pi$ for C_-, $0 \leq \arg(\zeta - z) \leq 2\pi$ for C_+,

$\zeta \neq z$, $z \in \mathbf{C}$, $\upsilon \in \mathbf{R}$, Γ: *Gamma function.*

Then $(f)_\upsilon$ is the fractional differintegration of arbitrary order υ (derivatives of order υ for $\upsilon > 0$, and integrals of order $-\upsilon$ for $\upsilon < 0$), with respect to z, of the function f, if $|(f)_\upsilon| < \infty$.

B. The set $\mathfrak{F}$

We call the function $f = f(z)$ such that $|f_\upsilon| < \infty$ in D as fractional differintegrable function by arbitrary order υ and denote the set of them with notation $\mathfrak{F} = \{f \,|\, |f_\upsilon| < \infty, \upsilon \in \mathbf{R}\}$. Then we have $|f_\upsilon| < \infty \Leftrightarrow f \in \mathfrak{F}$ *(in D).*

In order to discuss our main results, we need the following lemmas.

C. Lemmas

Lemma 1. ([1.Vol.1 P.41 Index law])

If $f(z)$ is an analytic and one-valued function and if $(f_\mu)_\upsilon$ and $(f_\upsilon)_\mu$ exist, then

$$(f_\mu)_\upsilon = f_{\mu+\upsilon} = (f_\upsilon)_\mu \qquad \text{for } f_\mu, f_\upsilon \neq 0,$$

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where $\mu,\ \upsilon \in \mathbf{R}$, $z \in \mathbf{C}$ and $\left|\dfrac{\Gamma(\mu+\upsilon+1)}{\Gamma(\mu+1)\Gamma(\upsilon+1)}\right| < \infty$.

Lemma 2. Let $U(z)$ and $V(z)$ are in $\mathfrak{F}$ respectively, then

(1) ([1.Vol.1 P.46 Generalized Leibniz's Rule])

$$(U \cdot V)_\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)} \cdot U_{\alpha-n} \cdot V_n$$

$$\left(\left|\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\right| < \infty\right),$$

where $\alpha \in \mathbf{R}$.

(2)

$$\left((z-a)^\beta\right)_\alpha = e^{-i\pi\alpha} \cdot \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \cdot (z-a)^{\beta-\alpha}$$

$$\left(\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right| < \infty\right),$$

where $z \neq a$, $\beta \neq 0$, $z \in \mathbf{C}$, $\alpha \in \mathbf{R}$.

(3) ([1.Vol.1 P.18])

$$(e^{az})_\alpha = a^\alpha \cdot e^{az} \qquad \text{where } a \neq 0,\ \alpha \in \mathbf{R}.$$

II. Main Results

First of all, we consider the equation of the form

$$\phi_2 - 2z\phi_1 + (\lambda-1)\phi = f \cdot e^{\frac{z^2}{2}}. \tag{2.1}$$

By using N^ν method, in 1998, Nishimoto ([10], P. 3) obtained a particular solution to this equation (2.1)

$$\phi = \left(\left((f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot e^{-z^2} \right)_{-1} \cdot e^{z^2} \right)_{\frac{-1-\lambda}{2}}. \tag{2.2}$$

Now, we discuss the generalized equation to (2.1) as follows.

Theorem 1. If $f \in \mathfrak{F}$ and $f_{\frac{\lambda-1}{2}} \neq 0$, then the generalized nonhomogeneous equation to (2.1)

$$\sum_{k=0}^{n} \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)} [\phi_{m-k} \cdot (z^n)_k - 2\phi_{m-k-1} \cdot (z^{n+1})_k] - 2\frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-n)\Gamma(n+2)} \phi_{m-n-2} \cdot (z^{n+1})_{n+1}$$

$$= f \cdot e^{\frac{z^2}{2}} \tag{2.3}$$

has a particular solution of the form

$$\phi = \left(e^{z^2} \cdot \left((f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot z^{-n} \cdot e^{-z^2} \right)_{-1} \right)_{1-m+\frac{1-\lambda}{2}}, \tag{2.4}$$

where $m \in \mathbf{Z}$, $n \in \mathbf{Z}^+ \cup \{0\}$, $z \in \mathbf{C}$, $\phi_0 = \phi = \phi(z)$, $f = f(z)$ is known, λ is a given constant.

Proof. Let

$$\phi = W_{\frac{1-\lambda}{2}} = (W(z))_{\frac{1-\lambda}{2}}, \tag{2.5}$$

hence

$$\phi_m = W_{\frac{1-\lambda}{2}+m}. \tag{2.6}$$

Substituting (2.6) into (2.3), we obtain

$$\sum_{k=0}^{n} \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)} [W_{\frac{1-\lambda}{2}+m-k} \cdot (z^n)_k - 2 \cdot W_{\frac{1-\lambda}{2}+m-k-1} \cdot (z^{n+1})_k] - 2\frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-n)\Gamma(n+2)}$$

$$\cdot W_{\frac{1-\lambda}{2}+m-n-2} \cdot (z^{n+1})_{n+1} = f \cdot e^{\frac{z^2}{2}}. \tag{2.7}$$

By using the Generalized Leibniz's Rule, we have

$$(W_m \cdot z^n)_{\frac{1-\lambda}{2}} = \sum_{k=0}^{n} \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)} \cdot W_{m+\frac{1-\lambda}{2}-k} \cdot (z^n)_k \tag{2.8}$$

and

$$(W_{m-1} \cdot z^{n+1})_{\frac{1-\lambda}{2}} = \sum_{k=0}^{n} \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)} W_{m+\frac{1-\lambda}{2}-k-1} \cdot (z^{n+1})_k + \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-n)\Gamma(n+2)} W_{m+\frac{1-\lambda}{2}-n-2} \cdot (z^{n+1})_{n+1}. \tag{2.9}$$

Considering (2.8) + (-2)(2.9), we have (by using (2.7))

$$(W_m \cdot z^n)_{\frac{1-\lambda}{2}} + (-2)\,(W_{m-1} \cdot z^{n+1})_{\frac{1-\lambda}{2}} = f \cdot e^{\frac{z^2}{2}}. \quad (2.10)$$

This implies

$$W_m \cdot z^n + (-2)W_{m-1} \cdot z^{n+1} = (f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}}. \quad (2.11)$$

Set

$$W_{m-1} = u = u(z), \quad (2.12)$$

we have

$$u_1 + u \cdot (-2z) = (f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot z^{-n}. \quad (2.13)$$

A particular solution to this linear first order ordinary differential equation is given by

$$u = e^{z^2} \cdot \left((f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot z^{-n} \cdot e^{-z^2} \right)_{-1} . \quad (2.14)$$

Thus (2.4) is obtained from (2.14), (2.12) and (2.5).

We can also verify directly, this (2.4) is a solution of (2.3).

Indeed, let

$$\phi = u_{1-m+\frac{1-\lambda}{2}}, \quad (2.15)$$

where

$$u = e^{z^2} \cdot \left((f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot z^{-n} \cdot e^{-z^2} \right)_{-1} , \quad (2.16)$$

we obtain

$$\text{L.H.S. of (2.3)} = \sum_{k=0}^{n} \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)} [u_{\frac{1-\lambda}{2}+1-k}$$

$$\cdot (z^n)_k - 2 \cdot u_{\frac{1-\lambda}{2}-k} \cdot (z^{n+1})_k]$$

$$-2\frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-n)\Gamma(n+2)} \cdot u_{\frac{1-\lambda}{2}-n-1}$$

$$\cdot (z^{n+1})_{n+1}$$

$$= \sum_{k=0}^{n} \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)} u_{\frac{1-\lambda}{2}+1-k}$$

$$\cdot (z^n)_k - 2\sum_{k=0}^{n+1} \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)} \cdot$$

$$u_{\frac{1-\lambda}{2}-k} \cdot (z^{n+1})_k$$

$$= (u_1 \cdot z^n - 2 \cdot u \cdot z^{n+1})_{\frac{1-\lambda}{2}}$$

$$= (2z \cdot e^{z^2} \cdot ((f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot z^{-n} \cdot e^{-z^2})_{-1}$$

$$\cdot z^n + e^{z^2} \cdot (f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot z^{-n} \cdot e^{-z^2} \cdot z^n$$

$$-2e^{z^2} \cdot ((f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot z^{-n} \cdot e^{-z^2})_{-1} \cdot z^{n+1})_{\frac{1-\lambda}{2}}$$

$$= ((f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}})_{\frac{1-\lambda}{2}}$$

$$= f \cdot e^{\frac{z^2}{2}} .$$

Next, we state a particular solution to the generalized nonhomogeneous Weber's equation.

Theorem 2. If $f \in \mathfrak{F}$ and $f_{\frac{\lambda-1}{2}} \neq 0$, then the generalized nonhomogeneous Weber's equation

$$\sum_{k=0}^{n} \frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)} [(\varphi \cdot e^{\frac{z^2}{2}})_{m-k}$$

$$\cdot (z^n)_k - 2(\varphi \cdot e^{\frac{z^2}{2}})_{m-k-1} \cdot (z^{n+1})_k]$$

$$-2\frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-n)\Gamma(n+2)} (\varphi \cdot e^{\frac{z^2}{2}})_{m-n-2} \cdot (z^{n+1})_{n+1}$$

$$= f \cdot e^{\frac{z^2}{2}} \quad (2.17)$$

has a particular solution of the form

$$\varphi = e^{-\frac{z^2}{2}} (e^{z^2} \cdot ((f \cdot e^{\frac{z^2}{2}})_{\frac{\lambda-1}{2}} \cdot z^{-n} \cdot e^{-z^2})_{-1})_{1-m+\frac{1-\lambda}{2}}, \quad (2.18)$$

where $m \in \mathbf{Z}$, $n \in \mathbf{Z}_0^+$, $z \in \mathbf{C}$, $\varphi_0 = \varphi = \varphi(z)$, $f = f(z)$ is known, λ is a given constant.

Proof. Letting $\phi = \varphi \cdot e^{\frac{z^2}{2}}$ in (2.3), then by using Theorem 1., we obtain that a particular solution to the generalized Weber equation (2.17) has the form

$$\varphi\cdot e^{\frac{z^2}{2}}=\left(e^{z^2}\left(e^{-z^2}\cdot\left(f\cdot e^{\frac{z^2}{2}}\right)_{\frac{\lambda-1}{2}}\cdot z^{-n}\right)_{-1}\right)_{1-m+\frac{1-\lambda}{2}}.$$

Thus, the theorem is proved.

Corollary 1. Weber equation

$$\varphi_2+\varphi\cdot(\lambda-z^2)=f\quad(\lambda\text{ is a constant})\tag{2.19}$$

has a particular solution of the form

$$\varphi=e^{-\frac{z^2}{2}}\left(e^{z^2}\cdot\left(\left(f\cdot e^{\frac{z^2}{2}}\right)_{\frac{\lambda-1}{2}}\cdot e^{-z^2}\right)_{-1}\right)_{-\frac{\lambda+1}{2}}.\tag{2.20}$$

Proof. Letting $m=2$ and $n=0$ in (2.17), then (2.17) becomes the Weber equation

$$\varphi_2+\varphi\cdot(\lambda-z^2)=f$$

and (2.18) becomes (2.20).

Note: Our result in Corollary 1 coincides with the result in Theorem 1 by Nishimoto in 1998 ([10]).

Theorem 3. The generalized homogeneous Weber's differential equation

$$\sum_{k=0}^{n}\frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-k+1)\Gamma(k+1)}\left[\left(\varphi\cdot e^{\frac{z^2}{2}}\right)_{m-k}\cdot(z^n)_k-2\left(\varphi\cdot e^{\frac{z^2}{2}}\right)_{m-k-1}\cdot(z^{n+1})_k\right]$$

$$-2\frac{\Gamma(\frac{1-\lambda}{2}+1)}{\Gamma(\frac{1-\lambda}{2}-n)\Gamma(n+2)}\left(\varphi\cdot e^{\frac{z^2}{2}}\right)_{m-n-2}\cdot(z^{n+1})_{n+1}$$

$$=0$$

has solutions of the form

$$\varphi=K\cdot e^{-\frac{z^2}{2}}\cdot\left(e^{z^2}\right)_{1-m+\frac{1-\lambda}{2}},$$

where $m\in\mathbf{Z}$, $n\in\mathbf{Z}_0^+$, $z\in\mathbf{C}$, $\varphi_0=\varphi=\varphi(z)$, λ is a given constant,and K is an arbitrary constant.

III. Example

Example. The third order nonhomogeneous Weber equation

$$\varphi_3\cdot z+\varphi_2\cdot(z^2+2)-\varphi_1\cdot(z^3+z)-\varphi\cdot(z^4+5z^2+2)$$

$$=e^{\frac{z^2}{2}}\cdot(4z^3+6z)\qquad(z\neq 0)\tag{3.1}$$

has a particular solution of the form

$$\varphi=e^{\frac{z^2}{2}}\cdot z.\tag{3.2}$$

Indeed, let $m=3$, $n=1$, $\lambda=-3$ and $=f\cdot e^{z^2/2}\cdot(4z^3+6z)$ in (2.17), then (2.17) becomes (3.1).

And, from (2.18), we obtain

$$\varphi=e^{-z^2/2}\left(e^{z^2}\cdot\left(\left(f\cdot e^{\frac{z^2}{2}}\right)_{\frac{\lambda-1}{2}}\cdot e^{-z^2}\right)_{-1}\right)_{-\frac{\lambda+1}{2}}.$$

$$=e^{\frac{z^2}{2}}\cdot\left(\left((4z^3+6z)\,e^{z^2}\right)_{-2}\cdot z^{-1}\cdot e^{-z^2}\right)_{-1}$$

$$=e^{\frac{z^2}{2}}\cdot z.$$

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Generalized Weber Equations via Fractional Calculus

利用分數微積分推廣 Weber Equation

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摘要

根據西本勝之教授分數微積分[1]，可獲得衆知的特殊二階微分方程式（如：Gauss，Legendre，Jacobi，Tchebycheff及Coulomb）的特解。

在參考文獻[2]-[7]中，一些學者再推廣上述各結果爲一般型並研論其特解。

最近，在1999年，於[8]，杜詩統教授等人又處理一般性的Associated Legendre，Euler，及Hermite的N階微分方程式的特解。在1998年，西本勝之教授，於[10]，採用N^{ν}方法，獲得在量子力學上著名的Weber方程式的解。

本篇論文主要是推廣1998年西本勝之教授的結果，及研論一般型Weber方程式的特解，並舉例説明。

關鍵詞：分數微積分，一般性Leibniz法則，一般型Weber方程式。