透過常態分配累積分配函數之有效率的近似公式分析違約機率

The Analyses of Default Probability by an Efficient Approximate Formula for the Cumulative Distribution Function of Standard Normal Distribution

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Contents

摘　要

本文提出一個效率的公式希望能較簡易且快速的計算出標準常態累積分配函數的趨近值，並將此公式結合財務上 Kealhofer–McQuown–Vasicek (KMV) 違約機率模型及 Merton 違約機率模型，藉此來分析違約機率在實務分析上的幾個相關議題，諸如：(1) 違約機率簡易效率的計算；(2) 給定違約機率下，公司資本槓桿與其資產風險之間的關係；(3) 當公司的資本結構或資產波動改變時，其違約機率有何變化。為說明公式如何實際應用，文中分別使用標準普爾 (Standard & Poor’s, S&P) 與臺灣企業信用風險指標 (Taiwan Corporate Credit Risk Index, TCRI) 的信用評等的資料進行數值分析。結果顯示，在違約機率的計算與分析上，與 Edous and Eidous (2018) 公式相比，本文之公式較具實務上的應用性。文中之數值分析結果亦提供以下幾個有用的訊息供市場參與者參考：(1) 當公司為最壞信用評等時，公司的資本槓桿率對其資產風險具最大的影響；(2) 資產風險對違約機率的影響比資本槓桿率對違約的機率影響還大；(3) 資本結構改變對最壞的信用評等公司的違約機率影響最大。在財務意義上，本文公式有助於市場參與者能易於瞭解違約機率相關變數的變化如何影響違約機率。由於本文所提出的公式易於實務上的應用，相信能協助市場監理者控管金融機構的違約風險，同時，亦能有助市場參與者有效率的管理具違約風險的複雜投資組合與進行最適投資決策。

關鍵詞：趨近值、常態分配、違約機率、資本槓桿率、資產風險
JEL 碼：C02, C60, G11

Abstract

This study supports an efficiency formula for more simply and more speedily calculating the approximation value of the cumulative distribution function of standard normal distribution. We combine this formula with the probability of default (PD) models, such as the Kealhofer–McQuown–Vasicek (KMV) model and Merton model, to analyze the following issues: (1) the simple and efficient calculations on PD; (2) the relationship between a firm's capital leverage and its asset risk under a given PD; and (3) the changes of PD when the firm changes its capital structure or its asset volatility. Numerical examples using Standard & Poor's (S&P) credit rating reports and Taiwan Corporate Credit Risk Index (TCRI) credit rating data illustrate the application of our formula. The results reveal that our formula owns a better applicability in practice for analyzing the PD compared with the formula shown in Edous and Eidous (2018). Our results also provide market participants the following useful financial information: (1) the influence of firm's capital leverage ratio on its asset risk has the largest effect for the worst credit quality; an increase on debt of asset (or asset volatility) induces a raise in the PD; (2) the influence of the asset risk on PD is larger than the influence of the capital leverage ratio on PD; and (3) if the firm in the worse credit rank, the change of its capital structure has great influence on its PD. On the financial applications, our formula can help market participants to easily understand how sensitive PD is to changes in its relevant variables. This not only can help market supervisors to manage the default risks for financial institutions, but also can help market participants to undertake the optimal investment decisions for the portfolios with default risks.

Keywords: Approximation, Normal Distribution, Default Probability, Capital Leverage Ratio, Asset Risk

JEL: C02, C60, G11

I. Introduction

In the theory of financial risk management, many market participants focus mainly on the estimation of extreme situations, such as the probability of default (PD). For evaluating the PD, there are two common methods: the structural-form approach and the reduced-form approach. The idea of pricing risky securities with PD first started with a structural-form approach (Merton, 1974; Black and Cox, 1976; Leland, 1994; Ambrose and Buttimer, 2000; Azevedo-Pereira, Newton and Paxson, 2003). This approach uses the market data to estimate the bank's asset value and its asset volatility, and then calculate the PD.

The reduced-form approach usually specifies the unpredictability of defaultable events as exogenous random variables that follow a Poisson distribution (Jarrow and Turnbull, 1995; Jarrow, 2001; Kau, Keenan and Smurov, 2004; Liao, Tsai and Chiang, 2008; Tsai, Liao and Chiang, 2009; Tsai and Chiang, 2012). The market information on hazard rate is used to calculate the PD. For the empirical analyses related with PD, several studies have employed Cox's proportional hazard model and either logistic or Poisson regression models to investigate which factors significantly affect the PD (e.g., Cox and Oakes, 1984; Schwartz and Torous, 1989, 1993; Lambrecht, Perraudin and Satchell, 2003). Our study discusses the PD with the structural-form approach.

In real world applications, the two models, belong to structural-form approach, are famous on the calculations of PD: the Kealhofer–McQuown–Vasicek (KMV) model and Merton's model. No matter using KMV model or Merton's model, the PD is calculated by the cumulative distribution function (CDF) for a standard normal distribution. Thus, market participants need a formula that can help them to efficiently and conveniently calculate the normal CDF for the tails of the normal distribution. Moreover, such formula is better to help them on the analyses of risk management. The main purpose of this paper is to provide a simple formula that can satisfy the requirement for market participants.

In traditional studies, because there is no explicit formula for the indefinite integral of a normal distribution, many scholars have advocated implicit formulas to approximate the normal CDF (Zelen and Severo, 1964; Marsaglia, 2004; Bowling, Khasawneh, Kaewkuekool and Cho, 2009; Soranzo and Epure, 2014). However, most of these formulas are difficult for practitioners to apply because they are quite complex. For overcoming such drawback, this study intends to provide a simple formula for calculating the normal CDF. The estimating error for the tails of the normal CDF distribution using our formula is quite small. Thus, our formula is very well suited for the analyzing extreme cases that financial risk theories focus on. For example, it can improve the calculation speed for the influence of various variables on the PD in response to changing economic conditions and in turn can help market participants to effectively measure their risks and undertake hedging strategies.

To illustrate the application of our model, we use the data published by Standard & Poor’s (S&P) Rating Corporation to provide a numerical example. For comparison purpose, we also adopt the data reported from Taiwan Corporate Credit Risks Index (TCRI), which is created by Taiwan Economic Journal (TEJ), to conduct the numerical analyses. Our analyses can show how market participants obtain more useful information, such as the influence of the asset-debt ratio and asset volatility on PD, by using our formula and the model of PD (such as KMV model and Merton’s model). Our formula does not only significantly increase calculation speed in the more complicated analyses of investment decisions, but it also enables market participants to better understand how sensitive PD is to changes in the relevant variables in numerical analyses.

The remainder of this paper is organized as follows. In Section II we describe the procedure for obtaining a simple formula that approximates the normal CDF. In Sections III and IV, we show how our formula is applied in obtaining values for PD. Finally, in Section V, we summarize our findings and offer suggestions for future research.

II. A Simple Formula for the Normal CDF

There are many scholars try to support implicit formulas to approximate the normal CDF (Zelen and Severo, 1964; Marsaglia, 2004; Bowling et al., 2009; Soranzo and Epure, 2014). In Appendix, we summarize some implicit formulas for the normal CDF that were provided in previous studies. For example, as shown in the second column of table in Appendix, the formula provided by Norton (1989) as follows:

$$N(x) \cong \frac{1}{2}\exp\left(-\frac{x^2 + 1.2x^8}{2}\right), \text{ for } 0 \le x \le 2.7, \tag{1}$$

where $N(x)$ is the function of normal CDF.

In financial theories, the estimations and the analyses for PD are important for financial supervisors who want to efficiently manage the financial institutions. It is also important for market participants who want to invest on the defaultable bonds and risky stocks for constructing a portfolio. In the uses of the structural-form approach, such as the KMV model and Merton's model, because the asset return is usually assumed to be normally distributed, the normal CDF is commonly appeared in the calculations for the option price and the PD. As well know, the normal CDF is expressed by an integral form. Thus, the value of normal CDF is usually obtained by the numerical method because there is no explicit formula for normal CDF. This will greatly restrict the calculations and analyses of PD for market participants. For example, if one only has a simple calculator which cannot perform the numerical calculation, (s)he cannot obtain the value of PD based on the integral form of normal CDF. Moreover, if one does not understand the Leibniz integral rule, (s)he may uneasy to understand the sensitivity analyses of PD. Thus, to support an explicit formula for normal CDF is important for financial supervisors and market participants.

However, as shown in Appendix, most of the explicit formulas for normal CDF are sophisticated in application. It derives our motivation to support a simple formula of normal CDF for financial supervisors and market participants who want to easily and quickly calculate the PD. Our formula can be effortlessly used to analyze the changes of PD due to the changes of influential factors, such as the equity/asset ratio and asset risk. Accordingly, it can help the financial supervisors to manage the bankruptcy risk of financial institutions and assist the market participants to construct a portfolio including defaultable securities and to hedge this portfolio.

Because the normal distribution is bell-shaped, here we show only the formula for the right side of the distribution, that is $N(x) > 0.5$, where $N(\bullet)$ is the traditional normal CDF. Our approximate formula is:

$$N(x) \approx 1 - a_i e^{-b_i x^2}, \text{ for } x \in N^{-1}(C_i), \tag{2}$$

where a_i and b_i, both positive values, are the parameters for the range $N^{-1}(C_i)$; C_i is the range of the probability; i is the index of this range; and $N^{-1}(\bullet)$ is the inverse of the normal CDF. Thus, $N^{-1}(C_i)$ is a domain of random variables corresponding to the probability range C_i.

Some researchers have advocated a different although similar formula for estimating the normal CDF (Chiani, Dardari and Simon, 2003; Olabiyi and Annamalai, 2012a, 2012b). Because they try to fit the normal CDF for the entire range of x, their formulas are quite complex and difficult to apply. We argue that for financial purposes the approximate formula needs not fit the normal CDF for the entire range of x, because most market participants focus only on the probabilities of extreme cases (e.g., likelihood of a huge loss, i.e., a tail probability). Thus, to simplify the formula for calculating the normal CDF, we divide x into intervals. Compared with the formulas used in traditional studies, our formula has the advantage of greater efficiency of calculation, making it more convenient for practical applications.

Here we provide a numerical example to illustrate the practical application of our formula. We divide the estimation range from 50% to 100% into 5% intervals.[1] In our example, there are ten classification ranges: C_1 = [0.50, 0.55), C_2 = [0.55, 0.60), C_3 = [0.60, 0.65), C_4 = [0.65, 0.70), C_5 = [0.70, 0.75), C_6 = [0.75, 0.80), C_7 = [0.80, 0.85), C_8 = [0.85, 0.90), C_9 = [0.90, 0.95), and C_{10} = [0.95, 1.00); [,) denotes a semi-open interval. For each range, we determine the critical values $N^{-1}(C_i)$ and estimate the corresponding parameter values a_i and b_i by minimizing the following sum square error:

$$\underset{a_i,b_i}{\mathrm{Min}} \sqrt{\sum_{x \in N^{-1}(C_i)} (N(x) - (1 - a_i e^{-b_i x^2}))^2}\,. \tag{3}$$

We let the value of x increase by 0.001 from its minimum to maximum value in each range of estimation. Table 1 shows the estimates of a_i and b_i for each range and the maximum error of the corresponding probability. The maximum error is defined as the maximum absolute value of the difference between the value of $N(x)$ and the value of $1 - a_i e^{-b_i x^2}$ for each x in the range. As shown in Table 1, for the ranges $N(x) \geq 70\%$, the maximum error is always less than 10^{-3} (0.1%). For example, for range 10, [0.95, 1.00), the estimated parameters are a_{10} = 0.2402 and b_{10} = 0.5870, and the maximum error is only

[1] Other size intervals could also be used.

9.2706×10^{-4} (0.093%). Figure 1(a)–1(f) show that for the ranges 5–10 the curves for $N(x)$ and $1 - a_i e^{-b_i x^2}$ are quite similar to one another.

As assumed in many financial theories, market participants care mainly about the risk associated with extreme events, i.e., $N(x) \geq 95\%$ or $N(x) \leq 5\%$. Thus, the results in range 10 demonstrate that our formula is very well suited for calculating the probability of extreme events in real world applications. For $N(x) \geq 95\%$, our formula can be expressed as:

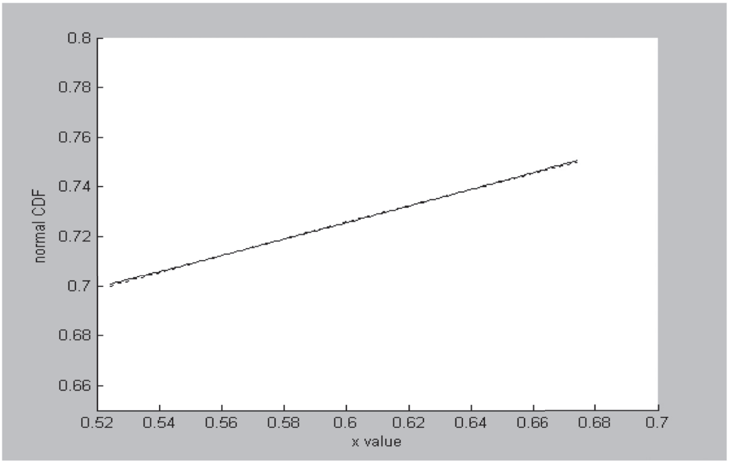
$$N(x) = P(X < x) \cong 1 - 0.2419e^{-0.5870x^2}, \text{ for } x \geq 1.645. \tag{4}$$

Table 1 Estimates of the Parameters and the Maximum Errors of Probability Using Our Formula

The range	Description	The estimated parameter a_i	The estimated parameter b_i	Maximum error
Range 1	50–55%	0.489107	6.062478	0.010893
Range 2	55–60%	0.464425	2.420855	0.002993
Range 3	60–65%	0.441023	1.584038	0.001612
Range 4	65–70%	0.418089	1.218315	0.001092
Range 5	70–75%	0.395398	1.013725	0.000797
Range 6	75–80%	0.372431	0.881580	0.000617
Range 7	80–85%	0.348449	0.787432	0.000516
Range 8	85–90%	0.322229	0.714738	0.000470
Range 9	90–95%	0.291310	0.653914	0.000474
Range 10	95–100%	0.240188	0.586986	0.000927
Range 11	99–100%	0.202436	0.557284	0.000081

The first column gives the estimated ranges. The second column gives the eleven ranges in our classification scheme. The third and fourth columns give the estimates of the parameters a_i and b_i. All parameters are estimated by minimizing the mean square error in each range, as described in Equation (2). The final column gives the maximum errors, defined as the maximum absolute value of the difference between $N(x)$ and $1 - a_i e^{-b_i x^2}$ for each x in the range.

(a) The estimated results in range 5



(b) The estimated results in range 6

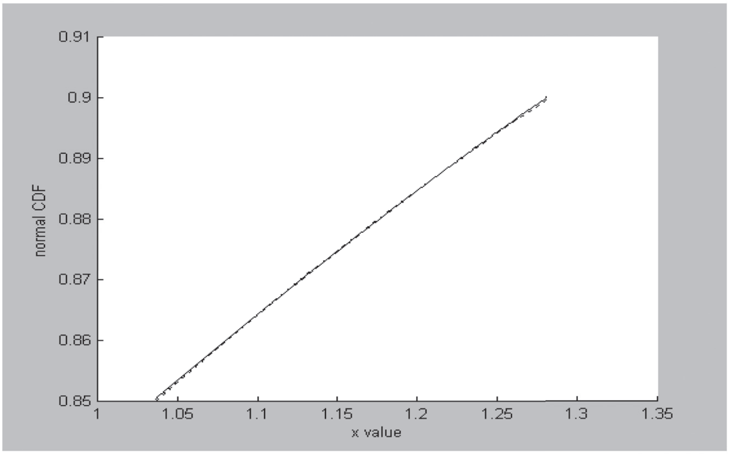
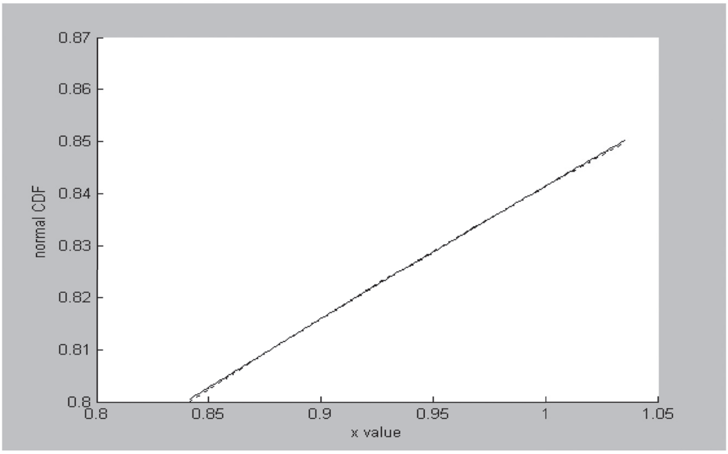


Figure 1 The True Normal CDF and Estimated Normal CDF in the Ranges 5–10 of Our Classification Scheme

(c) The estimated results in range 7



(d) The estimated results in range 8

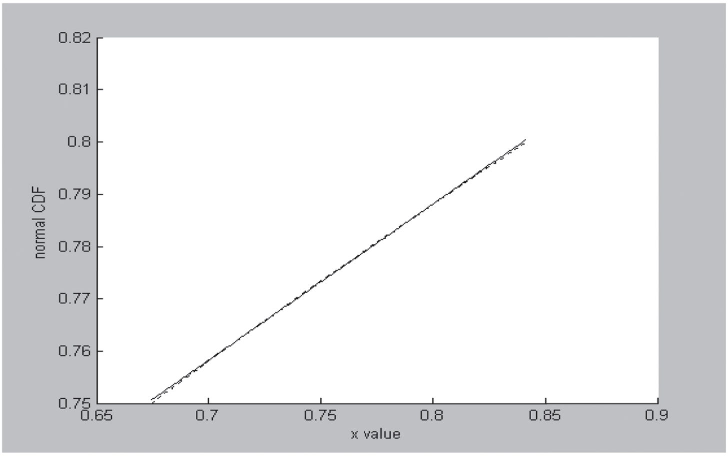
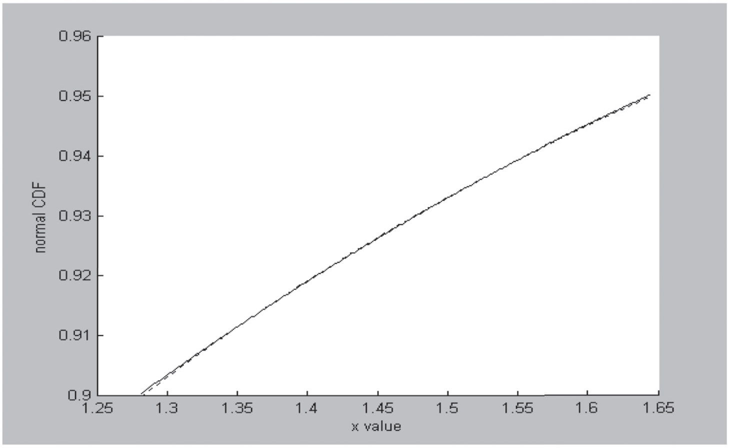


Figure 1 The True Normal CDF and Estimated Normal CDF in the Ranges 5–10 of Our Classification Scheme (continued)

(e) The estimated results in range 9



(f) The estimated results in range 10

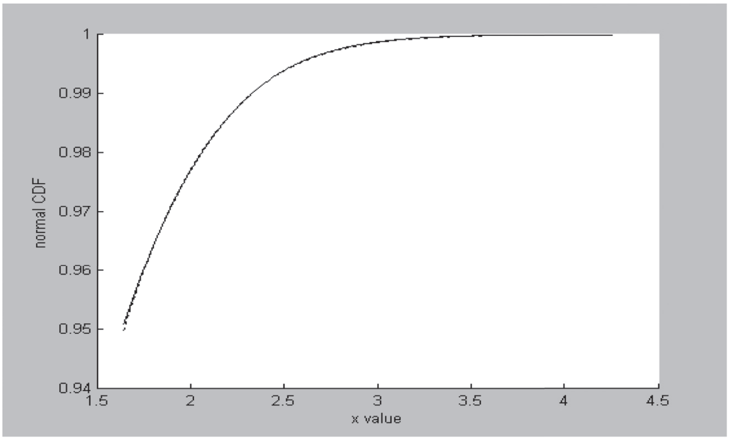


Figure 1 The True Normal CDF and Estimated Normal CDF in the Ranges 5–10 of Our Classification Scheme (continued)

The y-axis represents the normal CDFs. The x-axis is the x values. The solid line and the dotted line respectively represent the values of $N(x)$ and the values of $1 - a_i e^{-b_i x^2}$.

This formula is similar with the formula provided by Olabiyi and Annamalai (2012a, 2012b) that is shown in Appendix. Our formula of normal CDF seems quick simple in application. In financial risks management, it only needs a simpler formula for the efficiency calculation in extreme situations, such as PD. In the following section, we show how one can use this simple formula to effectively calculate the PD and also provide the numerical example to illustrate how one use it to undertake risk management for complicated portfolios.

III. An Application of the Formula to Obtain the PD

Analyzing the PD is essential for workers in the financial sector because the greatest losses for financial institutions come mainly from default. The KMV model, supported by KMV Corporation, is a famous model, used for measuring the PD by market participants. In this model, it calculates the distance to default (hereafter denoted as *DD*) and then estimate the PD. The value of *DD* is defined as the distance between the asset value and the default point (hereafter denoted as *DPT*). We denote the formula for *DD* as follows (see Crosbie and Bohn, 2003):

$$DD = \frac{V(0) - DPT}{V(0)\sigma_A} = \frac{\eta}{\sigma_A}, \tag{5}$$

where $V(0)$ is the asset value at the initial time; σ_A is the instantaneous standard deviation of a firm's asset return (i.e., asset risk); $\eta = \dfrac{V(0) - DPT(0)}{V(0)}$; $DPT(0) = CD(0) + \dfrac{1}{2}LD(0)$, is the default point at the initial time; $CD(0)$ is the short-term debt at the initial time; and $LD(0)$ is the long-term debt at the initial time.

Similar with the structural-form approach, when assuming the asset value to follow a log-normal distribution, the PD calculated from KMV model is expressed as

$$p = N(-DD), \tag{6}$$

where p is the PD. If the firm is still active, we usually have $DD > 0$. In KMV model, the auditing time, the point at which one can judge whether or not the firm is bankrupt, is usually assumed to be one year.

Next, we show how using our formula and KMV formula provides the interesting and useful information for the risk management. According to Equations (2), (5), and (6), we have

$$p = a_i e^{-b_i DD^2} \text{ and } DD = \sqrt{b_i^{-1}(\ln a_i - \ln p)}. \tag{7}$$

Using these formulas, one can deduce the relationship between DD and p. In Equation (7), a_i and b_i are known parameters. For example, if we let $p \leq 5\%$, one can derive the following formula to determine the relationship between DD and p, that is

$$DD = \sqrt{-2.4299 - 1.7036 \ln p}. \tag{8}$$

The formula in Equation (7) is also useful for analyzing other issues related to PD. For example, we can derive the partial derivative of p with respect to DD as follows:

$$\frac{\partial p}{\partial DD} = a_i e^{-b_i DD^2} \times (-2b_i DD) = -2pb_i DD. \tag{9}$$

We have $\frac{\partial p}{\partial DD} < 0$, which means that the relationship between DD and p is negative. Moreover, if the distance of default increases one unit, then the PD decreases by the amount of $-2pb_iDD$.

Furthermore, by using Equation (9), one can easily measure the influence of η (or σ_A) on default probability p. We have:

$$\frac{\partial p}{\partial \eta} = \frac{\partial p}{\partial DD}\frac{\partial DD}{\partial \eta} = -\frac{2pb_i DD}{\sigma_A} < 0, \text{ and} \tag{10}$$

$$\frac{\partial p}{\partial \sigma_A} = \frac{\partial p}{\partial DD}\frac{\partial DD}{\partial \sigma_A} = \frac{2pb_i DD^2}{\sigma_A} > 0. \tag{11}$$

Thus, an increase of η induces a decrease in the PD of a firm. Namely, if η increases one unit, then the PD reduces by the amounts $\frac{2pb_i DD}{\sigma_A}$. On contrast, an increase of σ_A induces a raise in the PD of a firm. That is, if the asset volatility increases one unit, the PD then increases by the amounts $\frac{2pb_i DD^2}{\sigma_A}$. The market participant can use these formulas to analyze the changes of PD when the firm changes its capital structure or its asset volatility. Moreover, the influence of increasing the asset risk (σ_A) on the PD is stronger (weaker) than that of increasing the equity-asset ratio on the PD if the *DD* is larger (less) than 1.

Market participants may want to know the relationship between a firm's capital leverage and its asset risk if the PD is known. Our formula can achieve this objective with great ease. For example, if $p = \bar{p}$, the relationship between η and σ_A is:

$$\sigma_A = \xi\eta, \tag{12}$$

where $\xi = (b_i^{-1}(\ln a_i - \ln\bar{p}))^{-\frac{1}{2}}$, a deterministic value. Using this formula can investigate the relationship between a firm's capital requirement and its asset risk based on the following equation

$$\frac{\partial \sigma_A}{\partial \eta} = \xi. \tag{13}$$

In other word, it is the linear relationship with slope ξ between the firm's equity-asset ratio and its asset risk given the PD in KMV model. If the equity of a firm decreases, the asset volatility (asset risk) should also decreases. When the equity-asset ratio decreases 1%, then the asset risk decreases ξ% for controlling the value of PD. The risk managers can use this formula to effectively undertake the risk management of investment portfolio for a firm.

We illustrate a simple example to show the application of our model. In financial markets, there are many credit rating companies, such as Moody's and S&P, that justify a firm's PD by giving it a specific credit grade (i.e., the default's transition probability matrix). Table 2 shows the statistic summary of PD from 1981 to 2014, yielded 34 sample number, in seven credit ranks (i.e., ranks AAA, AA, A, BBB, BB, B, and CCC) for global corporations. These data is published by S&P Rating Corporation in 2016.[2] According to Table 2, the average PDs for seven ranks (AAA, AA, A, BBB, BB, B, and CCC) are 0%, 0.0162%, 0.0624%, 0.2221%, 0.9535%, 4.5147%, and 23.6356% during the sample periods, respectively. Accordingly, the average PD is usually less than 1% for investment-grade firms (i.e., those with a credit rating higher than BBB). The maximum PD is only 1.02 (rank BBB) for investment-grade firms. In addition, even the PD for rank BB is also less than 1%. Thus our formula can be easily used to analyze the PD.

We also adopt the annual default rates reported from TCRI, which created by TEJ, to conduct the numerical analyses. TCRI comprises nine rating grades

2 Data source is from the 2014 Ratings Direct Report of S&P's Global Fixed Income Research and S&P's CreditPro®.

Table 2　Summary Statistics for the Seven Credit Ranks of Global Corporate Annual Default Rates Using S&P Data

	Annual default rate						
Statistics	AAA	AA	A	BBB	BB	B	CCC
Mean	0.0000	0.0162	0.0624	0.2221	0.9535	4.5147	23.6356
SD	0.0000	0.0695	0.1059	0.2591	1.0122	3.3119	11.8246
Min	0.00	0.00	0.00	0.00	0.00	0.25	0.00
Max	0.00	0.38	0.39	1.02	4.22	13.84	49.46

The global corporate annual default rates are reported by S&P's 2014 Ratings Direct Report. Our sample period from 1981 to 2014, yielded 34 sample number. The unit is percent. The symbols "Mean," "*SD*," "Min," and "Max" represent the average value, the standard deviation, the minimum value, and the maximum value for the PD of each credit rank, respectively.

except default (D) including: Grades 1–4 (low risk), Grades 5, 6 (middle risk), and Grades 7–9 (high risk). Table 3 shows the statistic summary of PD from 1999 to 2018, yielded 20 sample number, in nine credit ranks (i.e., ranks 1–9) for Taiwanese corporations. Because the PDs are zero for ranks 1–3, we only use the data in ranks 4–9 to perform the numerical analyses. The average PDs are 0.18%, 0.06%, 0.35%, 2.41%, 4.48%, and 11.21% for ranks 4–9, respectively. To be mentioned, the average PD on credit rank 4 is greater than that on credit rank 5 in the sample.

Given the above average PDs, it can find the corresponding values for a_i and b_i in Table 1. As using the S&P data, we conduct the following discussions, $a_{11} = 0.2024$ and $b_{11} = 0.5573$ are used to analyze the credit ranks AAA, AA, A, BBB, and BB; the estimated value $a_{10} = 0.2401$ and $b_{10} = 0.5870$ in the range 10 are used to analyze the credit rank B; and the estimated value $a_6 = 0.3724$ and $b_6 = 0.8816$ in the range 6 are used to analyze the credit rank CCC. When using the TCRI data, a_{11} and b_{11} are used to analyze the credit ranks 1–6; the estimated value a_{10} and b_{10} in the range 10 are used to analyze the credit ranks 7 and 8;

Table 3 Summary Statistics of Corporate Annual Default Rates for the Nine Credit Ranks Using TCRI Data

	Annual default rate						
Statistics	1–3	4	5	6	7	8	9
Mean	0.0000	0.1830	0.0560	0.3480	2.4095	4.4805	11.2060
SD	0.0000	0.3719	0.2238	0.7197	2.7327	3.3540	8.3104
Min	0.00	0.00	0.00	0.00	1.51	3.41	9.28
Max	0.00	1.00	1.00	3.00	9.00	11.00	32.00

The corporate annual default rates are reported by TCRI. Our sample period from 1999 to 2018, yielded 20 sample number. The unit is percent. The symbols "Mean," "*SD*," "Min," and "Max" represent the average value, the standard deviation, the minimum value, and the maximum value for the PD of each credit rank, respectively.

and the estimated value $a_8 = 0.3222$ and $b_8 = 0.7147$ in the range 8 are used to analyze the credit rank 9.

For comparison purposes, we show the estimates of PD from our formula and the formulas provided by the following two studies: Abderrahmane and Kamel (2016), and Edous and Eidous (2018). As shown in Appendix, Abderrahmane and Kamel (2016) had provided the following two approximate formulas:

$$N(x) = 1 - \frac{0.39894e^{-0.5078x^2}}{x + 0.79758e^{-0.4446x}}, \text{ for } 0 \leq x \leq 5, \text{ and} \tag{14}$$

$$N(x) = \frac{1}{2}(1 + \sqrt{1 - e^{-\frac{2}{\pi}x^2}}), \text{ for } 0 \leq x \leq 5. \tag{15}$$

Moreover, in Edous and Eidous (2018), the approximate formula is:

$$N(x) = 0.5(1 + \sqrt{1 - e^{-(0.647-0.021x)x^2}}). \tag{16}$$

For simplicity, here we define the formulas in Equations (14)–(16) as models 1–3, respectively. Also, our formula is defined as model 4.

In the estimates of PD, we firstly calculate the *DD* values by Equation (6) based on the actual average PDs reported by the credit rating company. After obtaining the *DD* values, we put them into models 1–4 for obtaining the estimates of PD. Finally, we calculate the root mean squared error (RMSE) between the actual average PDs and the estimated PDs for each model. The results are shown in Table 4. As shown in Table 4, when using the S&P data, the values of RMSE are 0.0170%, 0.1578%, 0.0080%, and 0.0089% for models 1–4 respectively. The RMSE is the smallest for model 3, the formula supported by Edous and Eidous (2018). In addition, the RMSE in our formula is better than the two formulas (i.e., models 1 and 2) shown in Abderrahmane and Kamel (2016). Table 4 also shows the results from these four models when using the TCRI data. The values of RMSE are 0.0161%, 0.2061%, 0.0169%, and 0.0085% for models 1–4, respectively. The RMSEs of our model is the smallest among four models. Accordingly, on the estimating accuracy of PD, our formula is better than that supported by Abderrahmane and Kamel (2016), and is not worse compared with the formula supported by Edous and Eidous (2018).

Table 5 shows the calculated value of ξ given the average PD in each rank using S&P data. Because the PD is zero for the rank AAA, we cannot analyze the relationship between η and σ_A. We thus analyze the relationship for all ranks except rank AAA. The calculated values of the slope (ξ) are 0.2796, 0.3105, 0.3514, 0.4271, 0.5912, and 1.3924 for the credit ranks AA, A, BBB, BB, B, and CCC. Using these values, we show the relationship between η and σ_A in Figure 2. Figure 2 tells us that there are linear relationships between η and σ_A in all credit ranks. The slope for the credit rank CCC of the firms is the highest among all credit ranks. It implies, given the certain PD, the increase in η for the credit rank CCC of the firms has the largest influence on their asset volatility.

Table 4 The Comparisons for the Estimates of PD under the Different Approximate Formulas of Normal CDF

Credit ranks	Mean of PD	Model 1	Model 2	Model 3	Model 4
		Using S&P data			
AA	0.0162	0.0150	0.0067	0.0155	0.0151
A	0.0624	0.0588	0.0329	0.0599	0.0609
BBB	0.2221	0.2132	0.1448	0.2161	0.2225
BB	0.9535	0.9329	0.7620	0.9451	0.9470
B	4.5147	4.4907	4.2007	4.5312	4.4944
CCC	23.6356	23.6608	23.5508	23.6335	23.6390
RMSE		0.0170	0.1578	0.0080	0.0089
		Using TCRI data			
4	0.1800	0.1723	0.1135	0.1747	0.1799
5	0.0600	0.0565	0.0315	0.0576	0.0585
6	0.3500	0.3381	0.2444	0.3425	0.3516
7	2.4100	2.3830	2.1290	2.4108	2.4185
8	4.4800	4.4559	4.1661	4.4962	4.4610
9	11.2100	11.2149	10.9630	11.2469	11.2100
RMSE		0.0161	0.2061	0.0169	0.0085

The unit is percent in this table. "RMSE" is the root mean of squared error. Models 1 and 2 are the formulas shown in Abderrahmane and Kamel (2016). They are $N(x) = 1 - \frac{0.39894e^{-0.5078x^2}}{x + 0.79758e^{-0.4446x}}$ for model 1 and $N(x) = \frac{1}{2}(1 + \sqrt{1 - e^{-\frac{2}{\pi}x^2}})$ for model 2. Model 3 is the formula shown in Edous and Eidous (2018). That is $N(x) = 0.5(1 + \sqrt{1 - e^{-(0.647-0.021x)x^2}})$. Model 4 is the formula supported by this study. It is $N(x) = 1 - a_i e^{-b_i x^2}$, as shown in Equation (2).

Table 5 Estimates of the Parameters and Influence of Parameters on PD Given Default Rate for Each Credit Rank Using S&P Data

	Credit rank					
Parameter	AA	A	BBB	BB	B	CCC
ξ	0.2796	0.3105	0.3514	0.4271	0.5912	1.3924
$\sigma_{A\|\eta=0.4}$	0.1118	0.1242	0.1406	0.1708	0.2365	0.5570
$\frac{\partial p}{\partial \eta}\|_{\eta=0.4}$	-0.0058	-0.0180	-0.0501	-0.1457	-0.3872	-0.5374
$\frac{\partial p}{\partial \sigma_A}\|_{\eta=0.4}$	0.0207	0.0581	0.1426	0.3411	0.6549	0.3859

ξ is the estimated slope for the relationship between η and σ_A as shown in Equation (12). $\sigma_{A|\eta=0.4}$, $\frac{\partial p}{\partial \eta}|_{\eta=0.4}$, and $\frac{\partial p}{\partial \sigma_A}|_{\eta=0.4}$ respectively mean the estimated asset volatility, the partial derivative of the PD with respect to η and the partial derivative of the PD with respect to the asset volatility (σ_A) when η to be 40%.

Our results are useful for market participants because they can use these results to estimate the risk when investing in the different credit rank of the firms. For example, as shown in Table 5, given the PD for all credit ranks by S&P credit rating, if initial equity-asset ratio is asked to be 40% (i.e., η = 0.4), the estimated asset volatility should be 0.1118, 0.1242, 0.1406, 0.1708, 0.2365, and 0.5570 for the credit ranks AA, A, BBB, BB, B, and CCC, respectively. It shows that the asset volatility for the worst credit rank of the firms is the highest among all credit ranks. The investors can calculate the portfolio frontier and determine the optimal investment decision based on the estimated risks.

Moreover, if initial η is asked to be 40%, according to Equation (10), we have $\frac{\partial p}{\partial \eta}$ equals to -0.0058, -0.0180, -0.0501, -0.1457, -0.3872, and -0.5374 for the credit ranks AA, A, BBB, BB, B, and CCC, respectively. Thus, for example, if a firm with η = 0.4 in the credit rank AA increases 1% of η, its PD will

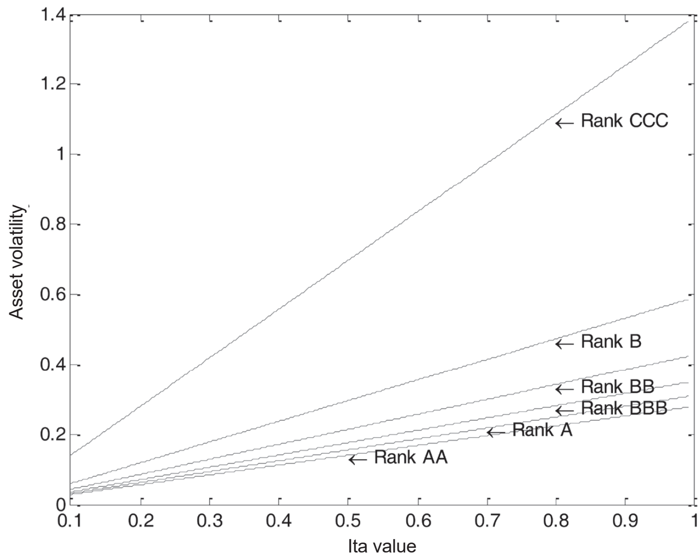


Figure 2 The Relationship between η and σ_A for Different Credit Ranks Using S&P Data

This figure shows the relationship between η and σ_A for different credit rank. The average PD of each rank is shown in Table 2.

decrease by 0.0058%. For the firm in the credit rank CCC, to improve its capital structure (i.e., to increase 1% of its η) will greatly decrease its PD by 0.3872%. It infers that if the firm in the better (worse) credit rank, the change of its capital structure has small (great) influence on its PD.

According to Equation (11), we have $\frac{\partial p}{\partial \sigma_A}$ equals to 0.0207, 0.0581, 0.1426, 0.3411, 0.6549, and 0.3859 for the ranks AA, A, BBB, BB, B, and CCC, respectively. Thus, for example, in rank AA, if a firm with η = 0.4 increases 1% of its asset risk, then its PD raises by 0.0207%. To be mentioned, the largest value is found in rank B. It implies that an increase in the asset risk has the largest influence on the PD of the credit rank B of the firm. In view of the above discussions, the influence of σ_A on PD is larger than the influence of η on PD for all ranks, except rank CCC.

Table 6 shows the calculated values of ξ, σ_A, $\frac{\partial p}{\partial \eta}$, and $\frac{\partial p}{\partial \sigma_A}$ given the average PD in each rank using TCRI data. The calculated values of the slope (ξ) are 0.3441, 0.3076, 0.3703, 0.5058, 0.5898, and 0.8226 for the credit ranks 4–9, respectively. Figure 3 also tells us that there are linear relationships between η and σ_A in all credit ranks. Moreover, the slope for the worst credit rank of the firms is the highest among all credit ranks. Thus, it implies a raise in η for the credit rank 9 of the firms has the largest impact on their asset volatility. To be mentioned, the slope for the credit rank 4 is the higher than that for the credit rank 5 since the average PD for credit rank 4 is greater than that for credit rank 5 in the sample.

If the initial equity-asset ratio is asked to be 40% (i.e., $\eta = 0.4$), the estimated σ_A should be 0.1376, 0.1230, 0.1481, 0.2023, 0.2359, 0.3290 for the credit ranks 4–9, respectively. It shows that σ_A for the credit rank 9 of the firms is the highest among all credit ranks. The values of $\frac{\partial p}{\partial \eta}$ equal to -0.0431, -0.0165, -0.0707, -0.2823, -0.3859, and -0.5918 for the credit ranks 4–9, respectively. This result tells us that if the firm in the better (worse) credit rank, the change of its capital structure has little (large) influence on its PD. Moreover, the values of $\frac{\partial p}{\partial \sigma_A}$ equal to 0.1251, 0.0536, 0.1909, 0.5581, 0.6543, and 0.7194 for the credit ranks 4–9, respectively. Accordingly, we can infer that the influences of σ_A on PD are larger than the influences of η on PD for all ranks. Figure 3 also shows the linear relationships between η and σ_A in all credit ranks. Our above results reveal that the findings are similar no matter using the S&P data or using TCRI data to conduct the numerical analyses.

Table 6 Estimates of the Parameters and Influence of Parameters on PD Given Default Rate for Each Credit Rank Using TCRI Data

	Credit rank					
Parameter	4	5	6	7	8	9
ξ	0.3441	0.3076	0.3703	0.5058	0.5898	0.8226
$\sigma_{A\|\eta=0.4}$	0.1376	0.1230	0.1481	0.2023	0.2359	0.3290
$\frac{\partial p}{\partial \eta}\|_{\eta=0.4}$	-0.0431	-0.0165	-0.0707	-0.2823	-0.3859	-0.5918
$\frac{\partial p}{\partial \sigma_A}\|_{\eta=0.4}$	0.1251	0.0536	0.1909	0.5581	0.6543	0.7194

ξ is the estimated slope for the relationship between η and σ_A as shown in Equation (12). $\sigma_{A|\eta=0.4}$, $\frac{\partial p}{\partial \eta}|_{\eta=0.4}$, and $\frac{\partial p}{\partial \sigma_A}|_{\eta=0.4}$, respectively mean the estimated asset volatility, the partial derivative of the PD with respect to η and the partial derivative of the PD with respect to the asset volatility (σ_A) when η to be 40%.

IV. The Analyses of PD When the Calculation of PD by Using the Merton's Model

The first methods proposed for calculating PD utilized a structural-form approach (Merton, 1974; Black and Cox, 1976; Leland, 1994). Under the assumptions that the debt is growth with riskless interest rate and the behavior of asset return is constructed under the risk neutral measure, PD in Merton (1974) is usually denoted as the follows:

$$p = N(-d_2), \tag{17}$$

where p is the PD and $d_2 = \dfrac{\ln\dfrac{V(0)}{B(0)} - \dfrac{1}{2}\sigma_A^2 T}{\sigma_A\sqrt{T}}$. If the firm is still active, we usually have $d_2 > 0$. $B(0)$ is the debt value at the initial time; and T is the

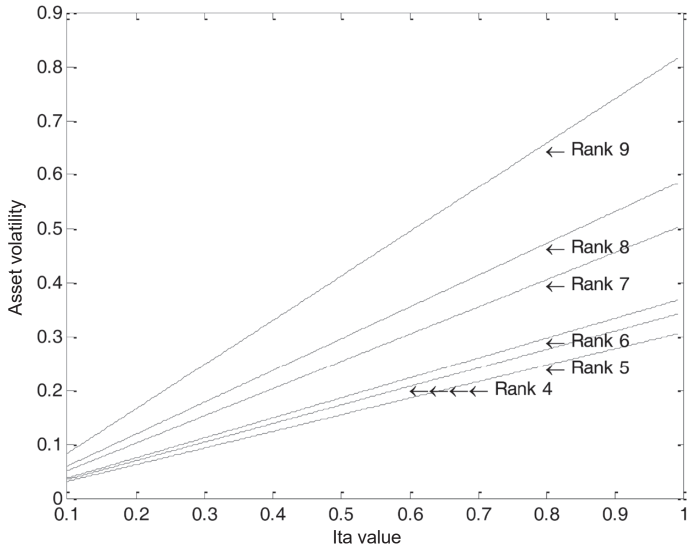


Figure 3 The Relationship between η and σ_A for Different Credit Ranks Using TCRI Data

This figure shows the relationship between η and σ_A for different credit rank. The average PD of each rank is shown in Table 4.

auditing time, the point at which one can judge whether or not the firm is bankrupt; $T = 1$ year is the specification generally used in traditional studies.

According to Equations (1) and (4), we have:

$$p = a_i e^{-b_i d_2^2} \text{ and } d_2 = \sqrt{b_i^{-1}(\ln a_i - \ln p)} \text{ for } d_2 \in N^{-1}(C_i). \tag{18}$$

Using these formulas, one can deduce the relationship between d_2 and p, because a_i and b_i are known parameters. For example, if we let $p \leq 5\%$, one can derive the following formula to determine the relationship between d_2 and p.

$$d_2 = \sqrt{-2.4299 - 1.7036 \ln p}. \tag{19}$$

If $p = \bar{p}$, the relationship between the asset/debt ratio $\frac{V(0)}{B(0)}$ and asset risk $\sigma\sqrt{T}$ is:

$$(\sigma_A\sqrt{T} + \xi^{-1})^2 = 2\ln\frac{V(0)}{B(0)} + \xi^{-2}. \tag{20}$$

This formula can be used to investigate the relationship between a firm's capital requirement and investment risk.

We also can derive the partial derivative of p with respect to d_2 as follows:

$$\frac{\partial p}{\partial d_2} = a_i e^{-b_i d_2^2} \times (-2b_i d_2) = -2p\sqrt{b_i(\ln a_i - \ln p)}. \tag{21}$$

In other words, $\frac{\partial p}{\partial d_2} \leq 0$, which means that the relationship between d_2 and p is negative. Furthermore, we have

$$\frac{\partial d_2}{\partial V(0)} = \frac{1}{V(0)\sigma_A\sqrt{T}} > 0, \tag{22}$$

$$\frac{\partial d_2}{\partial B(0)} = -\frac{1}{B(0)\sigma_A\sqrt{T}} < 0, \text{ and} \tag{23}$$

$$\frac{\partial d_2}{\partial(\sigma_A\sqrt{T})} = -\left(\frac{\ln\frac{V(0)}{B(0)}}{\sigma_A^2 T} + \frac{1}{2}\right) < 0. \tag{24}$$

Thus, by using Equations (21)–(24), one can easily determine the influence of $V(0)$, $B(0)$, and $\sigma\sqrt{T}$ on default probability p based on the results from the following:

$$\frac{\partial p}{\partial V(0)} = \frac{\partial p}{\partial d_2}\frac{\partial d_2}{\partial V(0)} = -\frac{2p\sqrt{b_i(\ln a_i - \ln p)}}{V(0)\sigma_A\sqrt{T}} \leq 0, \tag{25}$$

$$\frac{\partial p}{\partial B(0)} = \frac{\partial p}{\partial d_2}\frac{\partial d_2}{\partial B(0)} = -\frac{2p\sqrt{b_i(\ln a_i - \ln p)}}{B(0)\sigma_A\sqrt{T}} \geq 0, \text{ and} \tag{26}$$

$$\frac{\partial p}{\partial(\sigma_A\sqrt{T})} = \frac{\partial p}{\partial d_2}\frac{\partial d_2}{\partial(\sigma_A\sqrt{T})} = 2p\sqrt{b_i(\ln a_i - \ln p)}\left(\frac{\ln\frac{V(0)}{B(0)}}{\sigma_A^2 T} + \frac{1}{2}\right) \geq 0. \tag{27}$$

In other words, if the initial asset value increases one unit, then the PD decreases by the amount $\frac{2p\sqrt{b_i(\ln a_i - \ln p)}}{V(0)\sigma_A\sqrt{T}}$. On contrast, if the initial debt value and the asset volatility increase one unit, then the PDs increase by the amounts $\frac{2p\sqrt{b_i(\ln a_i - \ln p)}}{B(0)\sigma_A\sqrt{T}}$ and $2p\sqrt{b_i(\ln a_i - \ln p)}\left(\frac{\ln\frac{V(0)}{B(0)}}{\sigma_A^2 T} + \frac{1}{2}\right)$, respectively.

For the applications of KMV model, Crosbie and Bohn (2003) also supported another expression for *DD* by using Merton's theory. However, they assumed that the debt has no growth rate and the behavior of asset return is constructed under the Physical measure. Accordingly, the distance to default is expressed as follows (Crosbie and Bohn, 2003):

$$DD = \frac{\ln\frac{V(0)}{B(0)} + \mu T - \frac{1}{2}\sigma_A^2 T}{\sigma_A\sqrt{T}}, \tag{28}$$

where μ is the expected asset growth rate. Thus, we have:

$$\frac{\partial DD}{\partial(\sigma_A\sqrt{T})} = -\left(\frac{\ln\frac{V(0)}{B(0)} + \mu T}{\sigma_A^2 T} + \frac{1}{2}\right) < 0. \tag{29}$$

Based on Equation (11), we have:

$$\frac{\partial p}{\partial \sigma_A\sqrt{T}} = 2p\sqrt{b_i(\ln a_i - \ln p)}\left(\frac{\ln\frac{V(0)}{B(0)} + \mu T}{\sigma_A^2 T} + \frac{1}{2}\right) > 0. \tag{30}$$

Compared Equation (27) with Equation (30), the difference between KMV's model and Merton's model is $\frac{\mu}{\sigma_A^2}$. The reason for this difference mainly comes from the different assumptions on the debt's growth rate and the behavior of asset return.

V. Conclusions

In this study, we have presented a formula for easily calculating the CDF of a normal distribution. Our simple formula not only improves the efficiency of calculation, but it is also easy to apply in practice. Because the results from applying our formula reveal that the error of estimate for the tail probability is quite small, we decided to further apply it to investigate important financial issues associated with this tail probability, namely, PD.

Our formula is simpler on the estimating PD and has a better applicability in practice for analyzing the PD compared with the formula shown in Edous and Eidous (2018). According to our numerical example using S&P data, we summary the following findings: (1) given the certain PD, the increase in η for the credit rank CCC of the firms has the largest influence on their asset volatility; (2) an increase on debt of asset (or asset volatility) induces a raise in the PD for a firm; and if the firm in the better (worse) credit rank, the change of its capital structure has small (great) influence on its PD; and (3) the influence of the asset risk on PD is larger than the influence of the equity-asset ratio on PD for all ranks. The inferences are almost the same when using the TCRI data to conduct the numerical analyses.

The normal CDF is commonly used in analyses prescribed by many other financial theories, such as value-at-risk (VaR) and in formulas for determining pricing options. Our formula can significantly increase calculation speed in the more sophisticated investment analyses. Moreover, market participants can use it to investigate how the relevant variables influence PD. Thus, our formula should not only improve the management of the various risks of investment portfolios in response to changing economic conditions, but also can be profitably used in future studies to simplify the application of financial formulas that involve the normal CDF.

Appendix. The Approximate Formulas of the Normal CDF

Table A1 Summary the Approximate Formula of *N*(*x*)

Source	Approximate formula for *N*(*x*)
Norton (1989)	$N(x) \cong \frac{1}{2}\exp\left(-\frac{x^2+1.2x^8}{2}\right)$, $0 \leq x \leq 2.7$.
Johnson, Kotz and Balakrishnan (1994)	$N(x) \cong 1 - 0.5(a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4 + a_6x^5)^{-16}$, where $a_1 = 0.999998582$, $a_2 = 0.487385796$, $a_3 = 0.02109811045$, $a_4 = 0.003372948927$, $a_5 = -0.00005172897742$, $a_6 = 0.0000856957942$.
	$N(x) \cong 1 - \frac{1}{\sqrt{2\pi}}e^{(-0.5x^2-0.94x^{-2})}$, for $x \geq 5.5$.
	$N(x) \cong \frac{\exp\left(1.5976x(1+0.04417x^2)\right)}{1+\exp\left(1.5976x(1+0.04417x^2)\right)}$
	$N(x) \cong 1 - 0.5\exp\left(\frac{(83x+351)x+562}{\frac{703}{x}+165}\right)$
Bagby (1995)	$N(x) \cong 0.5 + 0.5\left(1 - \frac{1}{30}\left(7\exp\left(-\frac{x^2}{2}\right) + 16\exp\left(-x^2\left(2-\sqrt{2}\right)\right) + \left(7+\frac{\pi x^2}{2}\right)\exp(-x^2)\right)\right)^{0.5}$, for $x > 0$.
Waissi and Rossin (1996)	$N(x) \cong \frac{1}{1+\exp(-\sqrt{\pi}(\beta_1x^5+\beta_2x^3+\beta_3x))}$, $x \in [-8, 8]$, where $\beta_1 = -0.0004406$, $\beta_2 = 0.0418198$, $\beta_3 = 0.9$.
Bryc (2002)	$N(x) \cong \exp\left(-\frac{x^2}{2}\right) \times \frac{x+3.333}{\sqrt{2\pi}x^2+7.32x+6.666}$.
Chiani et al. (2003)	$erfc(\sqrt{x}) \leq \sum_{i=1}^{N} a_i e^{-b_i x}$, where $a_i = \frac{2(\theta_i - \theta_{i-1})}{\pi}$, $b_i = \frac{1}{\sin^2\theta_i}$, and $\theta_i = \frac{i\pi}{2N}$.
	For $N = 2$, $erfc(\sqrt{x}) \approx \frac{1}{6}e^{-x^2} + \frac{1}{2}e^{-\frac{4}{3}x^2}$.

Source	Approximate formula for *N*(*x*)
Marsaglia (2004)	$N(x) \cong 0.5 + (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \left(x + \frac{x^3}{3} + \frac{x^5}{3 \times 5} + \frac{x^7}{3 \times 5 \times 7} + \ldots \right)$, for $x > 0$.
Aludaat and Alodat (2008)	$N(x) \cong 0.5 + 0.5\sqrt{1 - e^{-\sqrt{\frac{\pi}{8}}x^2}}$, for $x > 0$.
Bowling et al. (2009)	$N(x) \cong \frac{1}{1 + \exp(-0.07056x^3 - 1.5976x)}$.
Olabiyi and Annamalai (2012a, 2012b)	$erfc(\sqrt{x}) \approx \sum_{k=1}^{N} a_k e^{-kbx}$, for $N = 1$, $a_1 = 0.4803$, $b = 1.1232$.
Soranzo and Epure (2012a)	$N(x) \cong \frac{1}{2} + \frac{1}{2}\sqrt{1 - \exp\left(-x^2 \frac{17 + x^2}{26.694 + 2x^2}\right)}$, for $x > 0$.
Soranzo and Epure (2012b)	$N(x) \cong \frac{1}{2} + \frac{1}{2}\sqrt{1 - \exp\left(\frac{-1.2735457x^2 - 0.0743968x^4}{2 + 0.1480931x^2 - 0.000258x^4}\right)}$, for $x > 0$.
Choudhury (2014)	$N(x) \cong 1 - \frac{1}{\sqrt{2\pi}} + \frac{e^{-\frac{x^2}{2}}}{0.226 + 0.64x + 0.33\sqrt{x^2 + 3}}$, for $x > 0$.
Soranzo and Epure (2014)	$N(x) \cong 2^{-22^{1-41^{x/10}}}$, for $x > 0$.
Winitzki (2014)	$N(x) \cong \frac{1}{2} + \frac{1}{2}\sqrt{1 - \exp\left(-\frac{x^2}{2} \frac{(4/\pi) + 0.147(x^2/2)}{1 + 0.147(x^2/2)}\right)}$, for $x > 0$.
Abderrahmane and Kamel (2016)	$N(x) = 1 - \frac{0.39894e^{-0.5078x^2}}{x + 0.79758e^{-0.4446x}}$, for $0 \le x \le 5$, and $N(x) = \frac{1}{2}\left(1 + \sqrt{1 - e^{-\frac{2}{\pi}x^2}}\right)$, for $0 \le x \le 5$.
Edous and Eidous (2018)	$N(x) = 0.5\left(1 + \sqrt{1 - e^{-(0.647 - 0.021x)x^2}}\right)$.

$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$, $N(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$, and $N(x)$ is the CDF of normal distribution.

References

Abderrahmane, M. and B. Kamel (2016), "Two News Approximations to Standard Normal Distribution Function," *Journal of Applied & Computational Mathematics*, 5(5), 1-2.

Aludaat, K. M. and M. T. Alodat (2008), "A Note on Approximating the Normal Distribution Function," *Applied Mathematical Sciences*, 2, 425-429.

Ambrose, B. W. and R. J. Buttimer (2000), "Embedded Options in the Mortgage Contract," *Journal of Real Estate Finance and Economics*, 21(2), 95-111.

Azevedo-Pereira, J. A., D. P. Newton and D. A. Paxson (2003), "Fixed-Rate Endowment Mortgage and Mortgage Indemnity Valuation," *The Journal of Real Estate Finance and Economics*, 26, 197-221.

Bagby, R. J. (1995), "Calculating Normal Probabilities," *The American Mathematical Monthly*, 102, 46-49.

Black, F. and J. C. Cox (1976), "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *The Journal of Finance*, 31, 351-367.

Bowling, S. R., M. T. Khasawneh, S. Kaewkuekool and B. R. Cho (2009), "A Logistic Approximation to the Cumulative Normal Distribution," *Journal of Industrial Engineering and Management*, 2(1), 114-127.

Bryc, W. (2002), "A Uniform Approximation to the Right Normal Tail Integral," *Applied Mathematics and Computation*, 127, 365-374.

Chiani, M., D. Dardari and M. K. Simon (2003), "New Exponential Bounds and Approximations for the Computation of Error Probability in Fading Channels," *IEEE Transactions on Wireless Communications*, 2, 840-845.

Choudhury, A. (2014), "A Simple Approximation to the Area under Standard Normal Curve," *Mathematics and Statistics*, 2, 147-149.

Crosbie, P. and J. Bohn (2003), "Modeling Default Risk," *Available at* https://www.moodysanalytics.com/-/media/whitepaper/before-2011/12-18-03-modeling-default-risk.pdf.

Cox, D. R. and D. Oakes (1984), *Analysis of Survival Data*, London, UK: Chapman and Hall.

Edous, M. and O. Eidous (2018), "A Simple Approximation for Normal Distribution Function," *Mathematics and Statistics*, 6(4), 47-49.

Jarrow, R. A. (2001), "Default Parameter Estimation Using Market Prices," *Financial Analysts Journal*, 57(5), 74-92.

Jarrow, R. A. and S. M. Turnbull (1995), "Pricing Options on Financial Securities Subject to Default Risk," *Journal of Finance*, 50, 481-523.

Johnson, N. L., S. Kotz and N. Balakrishnan (1994), *Continuous Univariate Distributions* (Vol. 1), 2nd Edition, New York: Wiley.

Kau, J. B., D. C. Keenan and A. A. Smurov (2004), "Reduced-Form Mortgage Valuation," *Working Paper*, University of Georgia.

Lambrecht, B. M., W. R. M. Perraudin and S. Satchell (2003), "Mortgage Default and Possession Under Recourse: A Competing Hazards Approach," *Journal of Money, Credit, and Banking*, 35, 425-442.

Leland, H. E. (1994), "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure," *The Journal of Finance*, 49, 1213-1252.

Liao, S.-L., M.-S. Tsai and S.-L. Chiang (2008), "Closed-Form Mortgage Valuation Using Reduced-Form Model," *Real Estate Economic*, 36, 313-347.

Marsaglia, G. (2004), "Evaluating the Normal Distribution," *Journal of Statistical Software*, 11(4), 1-11.

Merton, R. C. (1974), "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, 29, 449-470.

Norton, R. M. (1989), "Pocket Calculator Approximation for Areas under the Standard Normal Curve," *The American Statistician*, 43, 24-26.

Olabiyi, O. and A. Annamalai (2012a), "Invertible Exponential-Type Approximations for the Gaussian Probability Integral $Q(x)$ with Applications," *IEEE Wireless Communications Letters*, 1, 544-547.

Olabiyi, O. and A. Annamalai (2012b), "New Exponential-Type Approximations for the erfc(.) and erfc^{p}(.) Functions with Applications," In Institute of Electrical and Electronics Engineers (Ed.), *Proceeding of 2012 8th International Wireless Communications and Mobile Computing Conference*, (pp. 1221-1226), New York: Institute of Electrical and Electronics Engineers.

Schwartz, E. S. and W. N. Torous (1989), "Prepayment and the Valuation of Mortgage-Backed Securities," *Journal of Finance*, 44, 375-392.

Schwartz, E. S. and W. N. Torous (1993), "Mortgage Prepayment and Default Decisions: A Poisson Regression Approach," *Real Estate Economics*, 21, 431-449.

Soranzo, A. and E. Epure (2012a), "Practical Explicitly Invertible Approximation to 4 Decimals of Normal Cumulative Distribution Function Modifying Winitzki's Approximation of erf," *Available at* http://arxiv.org/abs/1211.6403.

Soranzo, A. and E. Epure (2012b), "Simply Explicitly Invertible Approximations to 4 Decimals of Error Function and Normal Cumulative Distribution Function," *Available at* http://arxiv.org/abs/1201.1320v1.

Soranzo, A. and E. Epure (2014), "Very Simply Explicitly Invertible Approximations of Normal Cumulative and Normal Quantile Function," *Applied Mathematical Sciences*, 8, 4323-4341.

Tsai, M.-S. and S.-L. Chiang (2012), "A Closed-Form Pricing Formula for Mortgages Incorporating Termination Hazard Rates and Recovery Rate as Correlated Stochastic Processes with Jump Components," *Journal of Derivatives*, 19(3), 49-68.

Tsai, M.-S., S.-L. Liao and S.-L. Chiang (2009), "Analyzing Yield, Duration and Convexity of Mortgage Loans under Prepayment and Default Risks," *Journal of Housing Economics*, 18, 92-103.

Waissi, G. and Rossin, D. F. (1996), "A Sigmoid Approximation to the Standard Normal Integral," *Applied Mathematics and Computation*, 77, 91-95.

Winitzki, S. (2014), "A Handy Approximation for the Error Function and Its

Inverse," *Available at* http://docs.google.com/viewer?a=v&pid=sites&srcid=ZGVmYXVsdGRvbWFpbnx3aW5pdHpraXxneDoxYTUzZTEzNWQwZjZlOWY2.

Zelen, M. and N. C. Severo (1964), "Probability Functions," In M. Abramowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, (pp. 925-996), New York: United States Department of Commerce and National Bureau of Standards.