Implementing Option Pricing Models when Asset Returns Follow an Autoregressive Moving Average Process

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ABSTRACT

Motivated by the empirical findings that asset returns or volatilities are predictable, this paper studies the pricing of European options on stock or volatility, the instantaneous changes of which depend upon an autoregressive moving average (ARMA) process. An ARMA process transforms to an MA process with new MA orders which depends on the observed time span under a risk-neutral probability measure. The pricing formula of an ARMA-type option is similar to that of Black and Scholes, except that the total volatility input depends upon the AR and MA parameters. Based on the results of numerical analyses, the option values are increasing functions of the levels of AR or MA parameters for all moneyness levels. Specifically, the AR effect is more significant than the MA effect.

1. INTRODUCTION

The pricing, hedging and risk management of derivatives is important because derivatives are now widely used to transfer risk in financial markets. At first, these options were priced and hedged using the classic Black–Scholes assumptions. In particular, it was assumed that stock price returns follow a geometric Brownian motion; however, it has been well-documented using empirical data that stock dynamics under the physical measure follow a more complicated process than the standard geometric Brownian motion. Hence, various extensions of the standard model have been proposed.

As pointed out by Lo and Wang (1995), there is now a substantial body of evidence that documents the predictability of financial asset returns. In addition to the mean-reverting model, the autoregressive moving average process (ARMA process) is one of the most popular models used to describe predictable financial asset returns. For example, Fama (1965) finds that the first-order autocorrelations of daily returns are positive for 23 of 30 Dow Jones Industrials. Fisher (1966) suggests that the autocorrelations of monthly returns on diversified portfolios are positive and larger than those for individual stocks. Gençay (1996) uses the daily Dow Jones Industrial Average Index from 1963 to 1988 to examine predictability of stock returns with buy-sell signals generated from the moving average rules. Lo and MacKinlay (1988) find that weekly returns on portfolios of NYSE stocks grouped according to size show positive autocorrelation. Similarly, Conrad and Kaul (1988) also present positive autocorrelations of Wednesday-to-Wednesday returns for size-grouped portfolios of stocks. Lo and MacKinlay (1990) report positive serial correlations in weekly returns for indices and portfolios and negative serial correlations for individual stocks. Chopra, Lakonishok and Ritter (1992), De Bondt and Thaler

(1985), Fama and French (1988), French and Roll (1986), Jegadeesh (1990), and Lehmann (1990) all find negatively serial correlations in returns of individual stocks or various portfolios.

It is also well known that the value of an option may depend on the log-price dynamics of underlying. The stock price process under Black–Scholes assumptions is a geometric Brownian motion, which implies an independently incremental property in stock returns. Distinguishing between the risk-neutral and true distributions of an option's underlying asset return process, Grundy (1991) shows that the Black-Scholes formula still holds, even though the underlying asset returns follow an Ornstein-Uhlenbeck (O-U) process. Along this line of research, Lo and Wang (1995) followed price options on an asset with a trending O-U process by using the Black-Scholes formula with an adjustment for predictability. They showed that as long as an Ito process with a constant diffusion coefficient describes the underlying asset's log-price dynamics, the Black-Scholes formula yields the correct option price regardless of the specification and arguments of the drift. Liao and Chen (2006) derive the closed-form formula for a European option on an asset, the returns of which follow a continuous-time type of first-order moving average process. The pricing formula of these options is similar to that of Black and Scholes, except for the volatility input. The first-order MA parameter is significant to option values even if the autocorrelation between asset returns is weak. However, there exist no studies concerning the extent to whether the Black-Scholes formula still holds when asset returns follow a generalized ARMA process. From the numerical analysis introduced later, the impact of AR parameters on the option value is more significant than is the MA effect, which shows that it is important to derive the pricing formula of European options with the instantaneous return of underlying assets following an ARMA process. Thus, the main goal herein is to fill-in this gap by introducing a continuous-time type of ARMA process, which is consistent with the findings in empirical studies, and to price European stock options by using the martingale pricing method.

The underlying asset's log-price dynamics in this study are similar to the discrete-time model used by Jokivuolle (1998) and the continuous-time one used by Liao and Chen (2006). Specifically, Jokivuolle (1998) values a European option on observed stock index returns which are specified as an infinite-order moving average process which is assumed to be different from true index returns. However, unlike Jokivuolle (1998), the process of asset returns in this study is a continuous-time type of ARMA process and the observed and true returns are identical, which is a common assumption in the martingale pricing methods. Furthermore, the MA(1) process used by Liao and Chen (2006) is the special case of the ARMA process, in which the AR order is zero and the MA order equals one. The MA order in Liao and Chen (2006) model is unchanged, but the ARMA process transforms to an MA process in which the new MA orders depend on the observed time span under risk-neutral probability measure.

2. DISCRIPTION OF THE MODEL

In this section, an ARMA process, which is composed of the AR effect on the drift term and the MA effect on the diffusion term of the instantaneous stock return, is first described; consequently, MA(1) process used by Liao and Chen (2006) is the special case of the ARMA process, in which the AR order is zero and the MA order equals one. The martingale measure, which makes the discounted stock price into a martingale, is also defined.

The ARMA process under physical measure becomes a MA process under martingale measure, the MA order of which depends upon the time span.

2.1 An ARMA Process of Instantaneous Asset Returns

Without loss of generality, this paper represents the underlying assets including dividends as *S*. The current time is t_0 and the expiration date of the options considered here is *T*. Since the stock returns of an autoregressive moving average process are a common finding in empirical studies, and as one actually observes a (negative) autocorrelation at very short lags in high-frequency return series, this paper introduces an ARMA process and assumes the dynamics of the instantaneous asset return as follows:

$$d\ln S_t = \mu\, dt + \sum_{i=1}^{p} \alpha_i\, d\ln S_{t-ih} + \sum_{j=0}^{q} \sigma \beta_j\, dW_{t-jh}^{P} \tag{1}$$

where *p* and *q*, respectively, denote the AR and MA orders, α_i and β_j are the AR and MA coefficients and $\beta_0 = 1$. μ is an arbitrary constant, $\sigma > 0$ is a constant volatility coefficient, $dt > 0$ is an infinitesimal time interval and $h > 0$ is a fixed, but arbitrary, small constant. In addition, W_t^P is a one-dimension standard Brownian motion defined in a naturally filtered probability space $\left(\Omega, \Im, P, (\Im_t)_{t\in[0,T]}\right)$ and dW_{t-ih}^P, $i = 1,\dots,N$, are the instantaneous increments of the standard Brownian motion at time $t-ih$. For empirical work, *h* is restricted by the frequency of historical data. It is convenient to assume that $T \in [t_N, t_{N+1})$, where $t_n = t_0 + nh$, $n = 0,\dots,N$. It is worth noting that Equation (1) reduces to the continuous-time MA(1) process in Liao and Chen (2006) when the AR and MA coefficients are all zero except for $\beta_0 = 1$ and $\beta_1 = \beta$.

To show the iterating procedure for an ARMA process, it follows from Equation (1) that

$$d\ln S_{t-\eta h} = \mu\, dt + \sum_{i=1}^{p} \alpha_i\, d\ln S_{t-(\eta+i)h} + \sum_{j=0}^{q} \sigma \beta_j\, dW_{t-(\eta+j)h}^{P}, \eta = 0,\dots,N \tag{2}$$

By substituting the right hand side of Equation (2) for $d\ln S_{t-ih}$ in Equation (1) step-by-step for $i = 1,\dots,n$, the dynamics of the stock price can be represented in the following Lemma.

Lemma 1. *Assume that the underlying stock price process S satisfies Equation (1). Given that* $y_1(-1) = 1$, $y_i(0) = \alpha_i$ *and* $\theta_j(0) = \beta_j$ *where* $i = 1,\dots,p$ *and* $j = 0,\dots,q$, *repeated substitution in Equation (1) n times yields*

$$d\ln S_t = \mu\left(\sum_{j=0}^{n} y_1(j-1)\right) dt + \sum_{j=1}^{p} y_j(n)\, d\ln S_{t-(n+j)h} + \sigma \sum_{j=0}^{n+q} \theta_j(n)\, dW_{t-jh}^{P} \tag{3}$$

where

$$y_j(k) = \alpha_j y_1(k-1) + 1_{(j<p)} y_{j+1}(k-1), \qquad for\ j = 1,\dots,p \tag{4}$$

$$\theta_j(k) = \begin{cases} \beta_{j-k}\, y_1(k-1) + 1_{(j<k+q)} \theta_j(k-1) & if\ \ j \ge k \\ \theta_j(j) & if\ \ j < k \end{cases} \tag{5}$$

Two important modeling issues concerning Equation (1) should be discussed. First, is the price process specified in Equation (1) used plausibly to represent security price fluctuations? Second, does the price process specified in Equation (1) rule out arbitrage opportunities? For the first issue, Harrison and Pliska (1981) demonstrate that as long as the discounted price process is a martingale under risk neutral probability measure *Q* equivalent to *P*, then the price process can be used to represent security price fluctuations. As for the second issue, it is well known that there are no arbitrage opportunities if and only if risk neutral probability measure *Q* exists. For expository purposes, the preceding condition will be checked in the next section.

2.2 Martingale Property of an ARMA Process

To price the financial derivatives written on a stock *S*, it is more convenient to have a risk-free security. Suppose the risk-free interest rate *r* is constant over the trading interval [0, *T*] and the saving account, denoted by *B*, is assumed to continuously compound in value at rate r; that is,

$$dB_u = r B_u du \ , \forall u \in [0, T]. \tag{6}$$

where equivalently, $B_t = e^{rt}$ with the usual convention that $B_0 = 1$. For $t \in [t_n, t_{n+1})$, the dynamics of the stock prices in (3) are equivalent to the following Itô integral equation:

$$\ln(S_t/S_{t_0}) = r(t-t_0) + Y_n(t_0,t) + Z_n^P(t_0,t), \ \ \forall t \in [t_n, t_{n+1}), n=0,\ldots,N \tag{7}$$

where

$$Y_n(t_0,t) = \left[\mu \sum_{i=0}^{n} y_1(i-1) - r\right](t-t_0) + \sum_{i=1}^{p} y_i(n)\int_{t_0}^{t} d\ln s_{u-(n+i)h}$$
$$+\sigma \sum_{i=n+1}^{n+q} \theta_i(n)\int_{t_0}^{t} dw_{u-ih}^{P} + \sigma \sum_{i=0}^{n} \theta_i(i)\int_{t_0}^{t_i} dw_{u-ih}^{P} \tag{8}$$

$$Z_n^x(t_0,t) = \sigma \sum_{i=0}^{n} \theta_i(i)\int_{t_i}^{t} dW_{u-ih}^{x}, \ \ x = P, Q, \ R \tag{9}$$

Conditioning on $\Im_{t_0}$, $d\ln s_{u-(n+i)h}$ and dw_{u-ih}^{P} in Equation (8) are the realized past increments of the log-price and Brownian motion, respectively. As the paths of stock price and the Brownian motion prior to the time t_0 are known, $Y_n(t_0,t)$ is $\Im_{t_0}$-measurable. In addition, rearranging $Z_n^P(t_0,t)$ as the sum of independent increments of Brownian motion, Equation (11) can be rewritten as follows:

$$Z_n^P(t_0,t) = \sigma \sum_{i=0}^{n} \theta_i(i)\int_{t_0}^{t-ih} dW_u^P = \sigma\left[\theta_0(0)\int_{t_0}^{t} dW_u^P + \theta_1(1)\int_{t_0}^{t-h} dW_u^P + \ldots + \theta_n(n)\int_{t_0}^{t-nh} dW_u^P\right]$$
$$= 1_{(n>0)}\left(\sigma \sum_{j=0}^{n-1}\left[\sum_{i=0}^{j}\theta_i(i)\right]\int_{t-(j+1)h}^{t-jh} dW_u^P\right) + \sigma\left[\sum_{i=0}^{n}\theta_i(i)\right]\int_{t_0}^{t-nh} dW_u^P \tag{10}$$

where 1_D is the indicator function, the value of which is 1 if *D* occurs and 0 otherwise. Therefore, the mean of geometric stock return at time *t* conditional on $\Im_{t_0}$ is

$$E_P\left(\ln(S_t/S_{t_0}) \,\middle|\, \Im_{t_0}\right) = r(t-t_0) + Y_n(t_0,t) \tag{11}$$

and the conditional variance obtained by using the independently incremental property of Brownian motion is given by

$$V_n(t_0,t) \equiv Var_P\left(\ln(S_t/S_{t_0}) \,\middle|\, \Im_{t_0}\right) = \sigma^2\left(1_{(n>0)}\sum_{j=0}^{n-1}\left(\sum_{i=0}^{j}\theta_i(i)\right)^2 h + \left(\sum_{i=0}^{n}\theta_i(i)\right)^2 (t-t_n)\right) \tag{12}$$

Based on the risk-neutral pricing theory, pricing the ARMA-type contingent claims is done under the martingale probability measure *Q* which makes the discounted stock price $\widetilde{S}_t = S_t/B_t$ into a Q-martingale, which can be represented as

$$E_Q\left(\widetilde{S}_u \,\middle|\, \Im_{t_0}\right) = \widetilde{S}_{t_0} \ , \forall u \in [t_0, T] \tag{13}$$

Based on both the dynamics of the stock price in Equation (9) and the definition of martingale measure *Q*, the transformation from probability measure *P* to *Q* is shown in the following Lemma.

Lemma 2. Assume that the dynamics of underlying stock price S satisfies Equation (1). The predictable process H defined as

$$H_n(x,y) = \int_x^y \varphi(z)dz, \text{ if } t_n \le x \le y \le t_{n+1}, n=0,\ldots,N \tag{14}$$

satisfies

$$H_n(t_n,t) = \frac{-\left(Y_n(t_0,t) + G_n(t_0,t) + \frac{1}{2}V_n(t_0,t)\right)}{\sigma}, \quad \forall t \in [t_n, t_{n+1}), \tag{15}$$

where

$$G_n(t_0,t) = \begin{cases} 1_{(n>1)}\sigma\sum_{j=0}^{n-1}\left(\sum_{i=0}^{j}\theta_i(i)\right)\left[H_{n-j-1}(t-(j+1)h,t_{n-j}) + H_{n-j}(t_{n-j},t-jh)\right] & \\ +\sigma\left[H_{n-1}(t-h,t_n) + \left(\sum_{i=0}^{n}\theta_i(i)\right)H_0(t_0,t-nh)\right] & \text{if } t\in[t_n,t_{n+1}),\ n\ge 1 \\ 0 & \text{if } t\in[t_0,t_1) \end{cases} \tag{16}$$

and φ *is an real- valued process. Then the process* W_t^Q, *which is given by the formula*

$$dW_t^Q = dW_t^P - \varphi(t)dt, \quad \forall t \in [t_0, T], \tag{17}$$

follows a one-dimensional Brownian motion on the probability space $(\Omega, \Im, Q)$.

In view of Lemma 2, the dynamics of the stock price defined in Equation (9) under martingale measure Q becomes

$$\ln(\widetilde{S}_t/\widetilde{S}_{t_0}) = Z_n^Q(t_0,t) - \frac{1}{2}V_n(t_0,t), \quad \forall t \in [t_n, t_{n+1}), n=0,\ldots,N \tag{18}$$

Since the quadratic variation of $Z_n^Q(t_0,t)$ over $[t_0, t]$ equals $V_n(t_0,t)$, as proved by Klebaner (1998, p.111), Equation (18) is the unique solution of the stochastic differential equation $dU_t = U_t\, dZ_n^Q(t_0,t)$, where $U_t = \widetilde{S}_t/\widetilde{S}_{t_0}$ and $U_{t_0} = 1$. Or equivalently, the dynamics of the stock price conditional on $\Im_{t_0}$ is as follows:

$$\frac{dS_t}{S_t} = r\,dt + \sum_{i=0}^{n}\theta_i(i)\,dW_{t-ih}^Q, \quad \forall t \in [t_n, t_{n+1}) \tag{19}$$

By virtue of (19), it suggests that the instantaneous stock returns, an ARMA process under physical measure P, follows a MA process under martingale measure Q, the MA order of which depends upon the integer part of time span $t - t_0$ divided by h. Besides, the new MA parameters $\theta_i(i)$ are affected by the original AR and MA parameters.

2.3 ARMA OPTION PRICING MODEL

Taking plain vanilla European call and put options as examples, the payoffs at the expiry date T are correspondingly $Max(S_T - K, 0)$ and $Max(K - S_T, 0)$, where K is the strike price. The time-t_0 values of European call options C_{t_0} and European put options P_{t_0} are given by

$$C_{t_0} = e^{-r(T-t_0)}E_Q\left[Max(S_T - K, 0)\,\middle|\,\Im_{t_0}\right], \quad P_{t_0} = e^{-r(T-t_0)}E_Q\left[Max(K - S_T, 0)\,\middle|\,\Im_{t_0}\right] \tag{20}$$

The pricing formulas of them are provided in the following Theorem.

Theorem 1. Assuming that the dynamics of the underlying stock prices are given by Equation (1), the *closed-form solutions for the ARMA(p, q)-type European options are as follows:*

$$C_{t_0} = S_{t_0}\,\Phi\left(d_{1N}\left(t_0,T\right)\right) - Ke^{-r(T-t_0)}\,\Phi\left(d_{2N}\left(t_0,T\right)\right) \quad (21)$$

$$P_{t_0} = Ke^{-r(T-t_0)}\,\Phi\left(-d_{2N}\left(t_0,T\right)\right) - S_{t_0}\,\Phi\left(-d_{1N}\left(t_0,T\right)\right) \quad (22)$$

where

$$d_{1z}(t,s) = \frac{\ln\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma_z^2(t,s)\right)(s-t)}{\sigma_z(t,s)\sqrt{s-t}},\quad d_{2z}(t,s) = d_{1z}(t,s) - \sigma_z(t,s)\sqrt{s-t}, \quad (23)$$

$$\sigma_z^2(t,s) = \frac{V_z(t,s)}{s-t}, \quad (24)$$

z is the integer part of $(s-t)/h$ for $s \ge t$ and $\Phi(\cdot)$ is the cumulative probability of an standard normal distribution. The proof of Theorem 1 is in Appendix C.

2.4 NUMERICAL ANALYSES OF ARMA-TYPE OPTIONS

To gauge the ARMA effect of stock return on the option's value, a one-month maturity European call option on stock is considered here, the instantaneous return of which follows ARMA(1,1) process. The initial stock price and volatility are set to 40 and 30%, respectively. The risk-free interest rate is 5%. Moreover, the ARMA effect on option price may depend on the strike price of the European call option. Hence, regarding the strike price, three target options are considered: an in-the-money (ITM), an at-the-money (ATM), and out-of-the-money (OTM) call option (K=90, 100, and 110, respectively).

[Insert Table 1 here]

Table 1 shows the ratio of one-month maturity ARMA(1,1)-type option prices in terms of BSM prices across different combinations of AR and MA parameters and various moneyness levels of the target option. Obviously, the option values increase as the level of AR or MA parameters increases for all moneyness levels. In particular, the AR effect on ARMA(1,1)-type option prices is more significant than the MA effect. To illustrate, in the case of $\alpha_1 = 0$, the absolute ratio increments for ITM, ATM, and OTM options are, respectively, 12.09% (108.55%-96.46%), 90.36% (145.68%-55.32%) and 273.79% (279.46%-5.67%) as β_1 changes from -0.5 to 0.5. In the case of $\beta_1 = 0$, however, the absolute ratio increments for ITM, ATM, and OTM options are, respectively, 22% (118.96%-96.96%), 118.94%(188.65%-69.71%), and 458.41%(481.51%-23.1%), as α_1 changes from -0.5 to 0.5. Thus, the AR effects dominate the MA effects for all moneyness levels.

In the case of $\alpha_1 + \beta_1 = 0$ (in the main diagonal for each Panel), the ARMA(1,1)- type option prices are equal to the BSM prices, which indicates that ignoring the impacts of an autocorrelation induced by AMRA-type process may not lead to a mispricing problem. However, in view of (14"), if $\alpha_1 + \beta_1 > 0$ ($\alpha_1 + \beta_1 < 0$) resulting in $\sigma_n^2(t_0,t) > \sigma^2$ ($\sigma_n^2(t_0,t) < \sigma^2$), the BSM prices undervalues (overprices) the ARMA(1,1)-type option prices. In particular, it is also observed that the impact of $\alpha_1 + \beta_1$ is asymmetric for all moneyness levels. For example, in the case of the ITM option, the ARMA(1,1)-type option price is higher than the BSM price by 43.67% when $\alpha_1 = \beta_1 = 0.5$ and is lower than the BSM price by 3.64% as $\alpha_1 = \beta_1 = -0.5$. Accordingly, in the case of lower values of the AR and MA parameters and the smaller strike prices, it is obvious that the difference between the ARMA(1,1)-type call prices and BSM prices is insignificant.

Given the AR or MA parameters, the absolute percentage differences between the BSM prices and the ARMA(1,1)-type option prices depend on the moneyness level. The absolute difference is maximized for ITM options and it is minimized for the OTM options. For example, the percentage absolute differences, which are correspondingly 3.64% (100%-96.36%), 58.65%(100%-41.35%) and 99.64%(100%-0.36%) for ITM, ATM and OTM options in the case of $\alpha_1 = \beta_1 = -0.5$, are likewise 1.4367, 2.7771 and 9.4161 times the BSM price for the ITM, ATM and OTM options when $\alpha_1 = \beta_1 = 0.5$, which indicates that the ARMA effects on option values depend not only on the level of AR and MA parameters but also the moneyness of the target option.

3. CONCLUSIONS

The evidence shows that daily, weekly and monthly returns are predictable from past returns. Motivated by the empirical findings that asset returns or volatilities are predictable, this paper studies the pricing of European options on the stock or volatility, the instantaneous logarithm increments of which depend upon an autoregressive moving average (ARMA) process. The dynamic for instantaneous stock return, an ARMA process under physical measure, transforms into an MA process with new MA orders depending on the observed time span under a risk-neutral probability measure. The pricing formula of an ARMA-type option is similar to BSM formula, except for the total volatility input depending upon the AR and MA parameters. Consequently, the implied volatility estimated from the BSM formula can be successfully interpreted as one calculated from an ARMA-type option formula. Specifically, this finding demonstrates that the BSM implied volatility is also valid even if the instantaneous stock returns follow an ARMA process.

In the absence of ARMA effects, the ARMA option pricing formula, indeed, reduces to the BSM pricing formula. When the AR and MA parameters are equal to zero (except for the first order MA coefficient), the ARMA option pricing formula reduces to the MA(1)-type option formula of Liao and Chen (2006). Furthermore, the ARMA-type option prices eventually converge to the BSM price when the time-to-maturity is approaching to zero. This result is in agreement with the assumption of Roll (1977), Duan (1995), Heston and Nandi (2000) and Liao and Chen (2006), where the option value with one period to expiration obeys the BSM formula. Based on the result of numerical analyses, the option values are increasing functions of the level of AR or MA parameters for all moneyness levels. Specifically, the AR effect is more significant than the MA effect.

APPENDIX A

The Proof of Lemma 1

Equation (1) can be proved by using mathematical induction. First, for the case of $n = 1$, by eliminating $d\ln S_{t-h}$ in Equation (1), the dynamics of the stock price can be represented as follows:

$$\begin{aligned} d\ln S_t &= \mu(1+\alpha_1)dt + \sum_{i=1}^{p-1}(\alpha_i\,\alpha_1 + \alpha_{i+1})d\ln S_{t-(i+1)h} + \alpha_p\alpha_1\, d\ln S_{t-(p+1)h} \\ &\quad + \sigma\, dW_t^P + \sum_{j=1}^{q}\left(\beta_{j-1}\,\alpha_1 + \beta_j\right)dW_{t-jh}^P + \beta_q\alpha_1\, dW_{t-(q+1)h}^P \\ &= \mu\left(\sum_{i=0}^{1} y_1(i-1)\right)dt + \sum_{i=1}^{1} y_i(1)\,d\ln S_{t-(i+1)h} + \sigma\sum_{i=0}^{q+1}\theta_i(1)\,dW_{t-ih}^P \end{aligned} \tag{A.1}$$

Consequently, Equation (3) holds for the case of $n = 1$. Assume that Equation (3) is valid for $n = k$, i.e.,

$$\begin{aligned} d\ln S_t &= \mu\left(\sum_{i=0}^{k} y_1(i-1)\right)dt + \sum_{i=1}^{p} y_i(k)\,d\ln S_{t-(k+i)h} + \sigma\sum_{i=0}^{k+q}\theta_i(k)dW_{t-ih}^P \\ &= \mu\left(\sum_{i=0}^{k} y_1(i-1)\right)dt + y_1(k)\,d\ln S_{t-(k+1)h} + \sum_{i=1}^{p-1} y_{i+1}(k)\,d\ln S_{t-(k+i+1)h} \\ &\quad + \sigma\sum_{i=0}^{k}\theta_i(i)\,dW_{t-ih}^P + \sigma\sum_{i=k+1}^{k+q}\theta_i(k)dW_{t-ih}^P \end{aligned} \tag{A.2}$$

Eliminating $d\ln S_{t-(k+1)h}$ in Equation (A.2), the dynamics of the stock price satisfies:

$$\begin{aligned} d\ln S_t &= \mu\left(\sum_{i=0}^{k} y_1(i-1)\right)dt + y_1(k)\left(\mu\, dt + \sum_{i=1}^{p}\alpha_i\, d\ln S_{t-(k+i+1)h} + \sum_{j=0}^{q}\sigma\beta_j\, dW_{t-(k+j+1)h}^P\right) \\ &\quad + \sum_{i=1}^{p-1} y_{i+1}(k)\,d\ln S_{t-(k+i+1)h} + \sigma\sum_{i=0}^{k}\theta_i(i)\,dW_{t-ih}^P + \sigma\sum_{i=k+l}^{k+q}\theta_i(k)dW_{t-ih}^P \\ &= \mu\left(\sum_{i=0}^{k+l} y_1(i-1)\right)dt + \sum_{i=1}^{p-1}\left[\alpha_i\, y_1(k) + y_{i+1}(k)\right] d\ln S_{t-(k+i+1)h} + \alpha_p y_1(k)\,d\ln S_{t-(k+p+1)h} \\ &\quad + \sigma\sum_{i=0}^{k}\theta_i(i)\,dW_{t-ih}^P + \sigma\sum_{j=k+1}^{k+q}\left[\beta_{j-(k+l)}y_1(k) + \theta_j(k)\right]dW_{t-jh}^P + \sigma\beta_q y_1(k)\, dW_{t-(k+q+1)h}^P \\ &= \mu\left(\sum_{i=0}^{k+1} y_1(i-1)\right)dt + \sum_{i=1}^{p} y_i(k+1)\, d\ln S_{t-(k+i+1)h} + \sigma\sum_{j=0}^{k}\theta_j(j)dW_{t-jh}^P + \sigma\sum_{j=k+1}^{k+q+1}\theta_j(k+1)\, dW_{t-jh}^P \\ &= \mu\left(\sum_{i=0}^{k+l} y_1(i-1)\right)dt + \sum_{i=1}^{p} y_i(k+1)\, d\ln S_{t-(k+i+1)h} + \sigma\sum_{j=0}^{(k+1)+q}\theta_j(k+1)\, dW_{t-jh}^P \end{aligned} \tag{A.3}$$

where $\theta_j(j) = \theta_j(k+1)$, for $j \le k$, is used in the last equation of Equation (A.3). Consequently, Equation (3) is also valid for the case of $n = k + 1$. This completes the proof of the Lemma.

APPENDIX B

The Proof of Lemma 2

By virtue of Equation (9), the dynamics of discounted stock prices satisfies:

$$\ln(\widetilde{S}_t/\widetilde{S}_{t_0}) = Y_n(t_0,t) + Z_n^P(t_0,t), \qquad \forall t \in [t_n, t_{n+1}) \tag{B.1}$$

Since $t - jh \in \left[t_{n-j}, t_{n-j+1}\right)$, it can be verified that

$$\int_{t_0}^{t-nh} dW_u^P = \int_{t_0}^{t-nh} dW_u^Q + \int_{t_0}^{t-nh} \varphi(z)\,dz = \int_{t_0}^{t-nh} dW_u^Q + H_0(t_0, t-nh) \tag{B.2}$$

$$\int_{t-(j+1)h}^{t-jh} dW_u^P = \int_{t-(j+1)h}^{t_{n-j}} dW_u^Q + \int_{t_{n-j}}^{t-jh} dW_u^Q + H_{n-j-1}\left(t-(j+1)h, t_{n-j}\right) + H_{n-j}\left(t_{n-j}, t-jh\right) \tag{B.3}$$

Substituting (B.2) and (B.3) into Equation (9), the dynamics of the discounted stock prices under the risk neutral measure Q are

$$\begin{aligned}\ln(\widetilde{S}_t/\widetilde{S}_{t_0}) &= Y_n(t_0,t) + Z_n^Q(t_0,t) + G_n(t_0,t) + \sigma H_n(t_0+nh,t) \\ &= \left[Y_n(t_0,t) + G_n(t_0,t) + \sigma H_n(t_0+nh,t) + \frac{1}{2}V_n(t_0,t)\right] + \left[Z_n^Q(t_0,t) - \frac{1}{2}V_n(t_0,t)\right]\end{aligned} \tag{B.4}$$

APPENDIX C

The Proof of Theorem 1

To carry out the proof of Theorem 1, Equation (22) is divided into two parts:

$$C_{t_0} = e^{-r(T-t_0)} E_Q\left[Max(S_T - K, 0) \,\middle|\, \Im_{t_0}\right] = A - B \tag{C.1}$$

where

$$A = e^{-r(T-t_0)} E_Q\left[S_T\, 1_{(S_T>K)} \,\middle|\, \Im_{t_0}\right] \tag{C.2}$$

and

$$B = K e^{-r(T-t_0)} E_Q\left[1_{(S_T>K)} \,\middle|\, \Im_{t_0}\right] \tag{C.3}$$

Under the risk neutral martingale measure Q , the stock price at time T equals

$$S_T = S_{t_0} \exp\left(r(T-t) + Z_N^Q(t_0,T) - \frac{1}{2}V_n(t_0,T) \right) \tag{C.4}$$

It is convenient to introduce an auxiliary probability measure Q_R on (Ω, F) by setting its Radon-Nikodym derivative as follows:

$$\xi_T^R = \frac{dQ_R}{dQ} = \exp\left(Z_N^Q(t_0,T) - \frac{1}{2}V_n(t_0,T) \right) \tag{C.5}$$

By virtue of Equation (12), $Z_N^Q(t_0,T)$ satisfies

$$Z_N^Q(t_0,T) = 1_{(N>0)}\left(\sigma \sum_{j=0}^{N-1}\left(\sum_{i=0}^{j}\theta_i(i)\right)\left(W_{t-jh}^Q - W_{t-(j+1)h}^Q\right)\right) + \sigma\left(\sum_{i=0}^{N}\theta_i(i)\right)\left(W_{t-Nh}^Q - W_{t_0}^Q\right) \tag{12'}$$

It follows from the Girsanov's theorem that the process W^R given by

$$W_v^R = W_v^Q - \sigma v \sum_{i=1}^{N} \theta_i(i), \ v \in \left[t_0, t - Nh\right) \tag{C.6}$$

$$W_v^R = W_v^Q - \sigma v \sum_{i=0}^{j} \theta_i(i), \ v \in \left[t-(j+1)h, t-jh\right), \ j = 0, \ldots, N-1 \tag{C.7}$$

is a standard Brownian motion under the probability measure Q_R. Therefore, $Z_n^Q(t_0,t)$ is equal to $Z_n^R(t_0,t) + V_n(t_0,T)$ and the dynamics of the stock price under the probability measure Q_R are

$$S_T = S_{t_0} \exp\left(r(T - t_0) + Z_N^R(t_0,T) + \frac{1}{2} V_n(t_0,T) \right) \tag{C.8}$$

Therefore, (C.2) can be rewritten as follows:

$$\begin{aligned}
A &= S_{t_0} E_{Q_R}\left(1_{(S_T > K)} \middle| \Im_{t_0} \right) = S_{t_0} P_{rob}{}^{Q_R}\left(S_T > K \middle| \Im_{t_0} \right) \\
&= S_{t_0} P_{rob}{}^{Q_R}\left(\ln S_{t_0} + r(T - t_0) + Z_N^R(t_0,T) + \frac{1}{2} V_N(t_0,T) > \ln K \middle| \Im_{t_0} \right) \\
&= S_{t_0} P_{rob}{}^{Q_R}\left(\frac{\ln \frac{S_{t_0}}{K} + r(T - t_0) + \frac{1}{2} V_N(t_0,T)}{\sqrt{V_N(t_0,T)}} \geq \frac{-Z_N^R(t_0,T)}{\sqrt{V_N(t_0,T)}} \right) \\
&= S_{t_0} \Phi\left(d_{1N}(t_0,T) \right)
\end{aligned}$$

(C.9)

Equation (C.3) can be proved along the similar pricing procedure. For the proof of (24), since the discounted stock price and the ARMA(p, q)-type European options are Q-martingale, (24) is derived by using the well-known put call parity $P_{t_0} = C_{t_0} + Ke^{-r(T-t_0)} - S_{t_0}$. This ends the derivation of the pricing formulas for ARMA-type European options.

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