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# Boundary Control of Subdivision Surfaces Using an Open Uniform Quadratic Subdivision Scheme*

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The ability to control boundary behaviors is critical for the subdivision surface to become a useful representation in computer aided geometric design. Previous methods achieve boundary control by extending the boundary vertices of the control mesh, or by applying different subdivision rules to the boundary edges and vertices. This paper presents an open uniform quadratic subdivision scheme derived from the subdivision of open uniform B-spline surfaces. For meshes with 4-sided boundary and corner faces, the cross-tangent along the boundary curves can also be derived. The proposed scheme leads to a straightforward boundary control that enables two subdivision surfaces to be joined with $C^0$ and $C^1$-continuity.

***Keywords:*** geometric modeling, subdivision surfaces, boundary control, parametric surfaces, blending surfaces

## 1. INTRODUCTION

Although the non-uniform rational B-spline (NURBS) surface has become an industry standard in computer graphics and CAD/CAM systems, surfaces of arbitrary topology can not be represented by using a single NURBS surface, and therefore there has been considerable interest in the subdivision surface since it was first proposed in 1978 [1, 2]. Some important properties of the subdivision surface, such as the continuity and behavior near the extraordinary points, have been extensively studied; see for example [2-6]. In [7, 8] the subdivision scheme is used to reconstruct the smooth spline surfaces from unorganized points. Several problems need to be further studied before the subdivision surface becomes useful in computer aided geometric design (CAGD); for example, the boundary control of the subdivision surface is investigated in [9, 10]. Since each face of the control mesh converges toward its centroid for the quadratic subdivision surfaces, the boundaries of the subdivision surface are not confined to be on the boundaries defined by the original control mesh. Nasri [9] solved this problem for quadratic subdivision by extending the boundary of the original control mesh so that the boundary curves of the limiting subdivision surface become the B-spline curves defined by the original boundary edges. At each subdivision step, however, the boundary curves defined by the boundary edges are not the boundary curves of the limiting surface. This may cause problems when we join two subdivision surfaces with intermediate control

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meshes, since two connected surfaces may have to overlap at any finite step of subdivision. In [11, 12] the polygonal complexes are used to interpolate the predefined boundary or interior curves. In [13, 14], different subdivision rules are applied to some edges or vertices of the control mesh, and the shape or boundary of the subdivision surface can be controlled. In [15-17] a combined subdivision scheme is proposed to control the boundary of the subdivision surface, which can then be used to fill an N-sided hole.

In this paper we propose an open uniform quadratic subdivision scheme, which is derived from the subdivision of an open uniform B-spline surface. With this scheme, the boundary curves of the subdivision surface at each subdivision step are in B-spline form and are always the same boundary curves of the limiting surface. This is useful when intermediate polygons are needed to render the subdivision surface. The subdivision surface generated by this scheme is also shown to have $C^1$ continuity. We also propose the derivation of the cross-tangent of the subdivision surfaces for control meshes having 4-sided corner and boundary faces. This yields a straightforward boundary control that is capable of smoothly joining two subdivision surfaces or a subdivision surface and a B-spline surface.

This paper is organized as follows. Section 2 reviews the subdivision surface and its boundary control. Section 3 describes the subdivision scheme for uniform quadratic B-spline patches. Section 4 describes the proposed open uniform subdivision scheme for control meshes of restricted topology. Section 5 gives some examples. Finally, in section 6 we present conclusions and give some future work.

## 2. SUBDIVISION SURFACE AND ITS BOUNDARY CONTROL

The subdivision surface is defined by recursively subdividing a mesh of arbitrary topology. At each step, more vertices and smaller faces are created, and the surface is defined as the limit of the subdivision process. For a rectangular control mesh, after several steps, the new control mesh consists of faces arranged in a regular lattice, except at some fixed number of extraordinary points, where the number of incident edges is not four. Since the limiting surface derived from each rectangular mesh is a B-spline surface, the problem of smoothness arises only inside the non-quadrilateral faces. The continuity conditions near the extraordinary points has been analyzed using the Fourier transform [2], a matrix approach [3], and characteristic map [5, 6]. In this section, we first review the formulations for subdivision surfaces in section 2.1, then we review the boundary control scheme for the quadratic subdivision surfaces in section 2.2.

### 2.1 Subdivision Schemes

There are mainly two subdivision schemes: cubic and quadratic subdivision [1, 2].

• The cubic subdivision scheme:

At each subdivision three distinguishable classes of points are generated: (see Fig. 1 (a)).

1. The *new face points F'* of a face *F*, which is the weighted average of all the old points defining the face *F*.

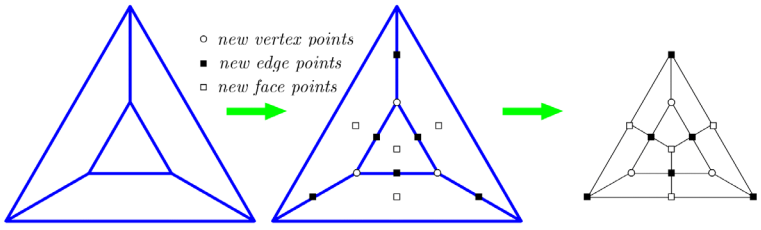
2. The *new edge points E'* of an edge *E*, which is the average of the end points of edge *E* and the two new face points of the faces sharing edge *E*.
3. The *new vertex points V'* of a vertex *V*, which is derived using the weighted average

$$V' = \frac{Q}{n} + \frac{2R}{n} + \frac{V(n-3)}{n}, \tag{1}$$

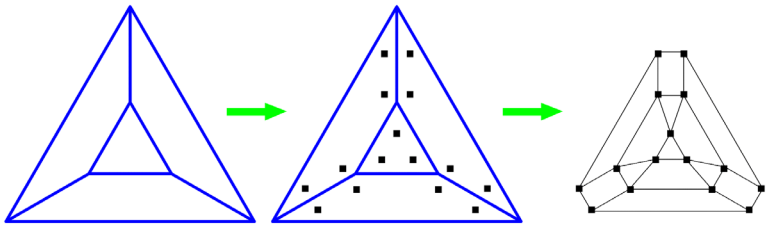
where *n* is the number of edges incident to *V*, *Q* is the weighted average of the new face points of faces sharing *V*, and *R* is the weighted average of the midpoints of edges incident to *V*.

After these points have been computed, new edges are formed by (see Fig. 1 (a))

– connecting each new face point to the new edge points of the edges defining the old face.
– connecting each new vertex point to the new edge points of all old edges incident on the old vertex point.




(a) Cubic subdivision.



(b) Quadratic subdivision.

Fig. 1. Cubic and quadratic subdivision.

Note that after one step of subdivision, the control mesh consists only of quadrilateral faces. In [3] a general form for the new vertex points was proposed and it is shown that the subdivision scheme using Eq. (1) generates limiting surfaces with $C^1$-continuity.

• The quadratic subdivision scheme:

At each subdivision, *n* new vertices are generated for each *n*-sided face *F* on the control mesh. The new control mesh is constructed by connecting these vertices across the old edges; see Fig. 1 (b). Each new vertex can be calculated from the vertices defining the face *F* using the general formulation

$$V_i' = \sum_{j=1}^{n} \alpha_{ij} V_j$$

where $V_j$, $j = 1, \ldots, n$ are the vertices of the old face $F$, and $V_i'$ is the new vertex corresponding to $V_i$. In [1] the following are used:

$$\begin{aligned} \alpha_{ij} &= (4n + 2)/8n \quad && |i - j| = 0 \\ \alpha_{ij} &= (n + 2)/8n && |i - j| = 1 \\ \alpha_{ij} &= 2/8n && |i - j| > 1. \end{aligned} \tag{2}$$

In [2], it is shown that the limiting subdivision surface defined by Eq. (2) is not guaranteed to have $C^1$-continuity, while the new formulation

$$\begin{aligned} \alpha_{ij} &= (n + 5)/4n && i = j \\ \alpha_{ij} &= (3 + 2\cos(2\pi(i - j)/n)/4n && i \neq j \end{aligned} \tag{3}$$

generates a $C^1$-continuous limiting subdivision surface, but gives discontinuity of exactly the second derivative under all circumstances.

### 2.2 Boundary Control for Quadratic Subdivision Surfaces

In [9, 10] the boundary control of the quadratic subdivision surface is achieved by extending the corner and boundary vertices of the given mesh. For a 4-sided boundary face, each interior vertex immediately adjacent to a boundary vertex is reflected with respect to the boundary vertex. In Fig. 2 (a), for instance, vertices $P_3$ and $P_6$ are reflected with respect to $P_4$ and $P_5$ yielding $P'_4$ and $P'_5$, respectively. The new face is still four-sided but defined by vertices $P'_4$, $P'_5$, $P_3$, and $P_6$. The quadratic subdivision on this 4-sided face will converge to the centroid $G = \frac{1}{2}(P_4 + P_5)$.

For a 4-sided corner face, the desired corner of the subdivision surface is at the corner vertex of the face. As shown in Fig. 2 (b), suppose that the two boundary vertices of the corner face, $P_4$ and $P_2$, are reflected to obtain $P'_4$ and $P'_2$. The new corner vertex $P'_1$ can be chosen so that $P_1$ becomes the centroid of the new corner face formed by $P'_1$, $P'_4$, $P_3$, and $P'_2$. Hence $P'_1 = 4P_1 - 2P_2 - 2P_4 + P_3$.

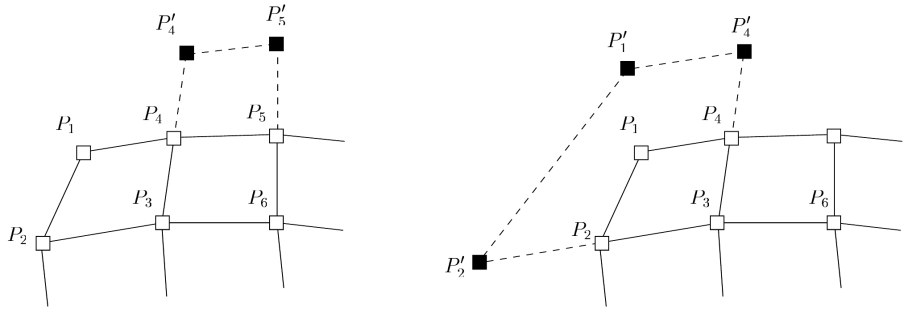



Fig. 2. Boundary control by extending the control mesh.

The surface's boundary derived from the new control mesh using quadratic subdivision converges to a curve passing the midpoint of each boundary edge and the corner vertices. As a result, each boundary curve of the limiting surface is an open uniform B-spline curve defined by the respective boundary edges and corner vertices of the original control mesh. The boundary control for the general control meshes with restricted topology, in which each boundary vertex corresponds to only one nonboundary edge, is achieved in a similar manner [10]. In [11, 12] Nasri shows that the limit curve of the symmetric polygonal complex is predictable, and hence can be used to interpolate predefined boundary or interior curves. This method provides a solution to generate creases on the subdivision surface without modification of the subdivision rules.

In [14] a tagging scheme that applies different subdivision rules is used to sharpen edges or vertices to interpolate predefined curves or vertices. In [13] Habib also uses a different subdivision scheme which places neighboring vertices at the midpoint of two parents to interpolate the boundary vertices.

While the cubic scheme can generate limiting surfaces with continuity of $C^1$ or above, the quadratic scheme provides only $C^1$-continuity. Both schemes are capable of joining two or more surfaces smoothly. We found that the boundary curves of subdivision surfaces can be the same as the open uniform B-spline curves defined by corresponding boundary control points with our modified quadratic subdivision scheme. The difference between our approach and methods proposed in [13] is that our's scheme is the general subdivision scheme of an open uniform B-spline surface, and our subdivision rules for interior vertices of boundary faces are different. Under our scheme the boundary curves of the subdivision surface at each subdivision step are always the same boundary B-spline curves of the limiting surface. This is useful when intermediate polygons are used in rendering the subdivision surface, especially when joining one subdivision surface with another or with B-spline surface. Furthermore, it is usually desirable to be able to control the cross-tangent along the boundary curves of the surface during subdivision iteration, for example, when approximating a trimmed hole on a parametric surface. We also show that the proposed subdivision surface is $C^1$ continuous and how the cross-tangent along the boundary of the surface can be derived and controlled.

## 3. OPEN UNIFORM QUADRATIC SUBDIVISION SCHEME FOR RECTANGULAR MESHES

In this section we derive the open uniform quadratic subdivision scheme for rectangular control meshes. In section 3.1, we describe the subdivision scheme for the rectangular control mesh of a uniform B-spline patch in a similar way to the derivation of the cubic subdivision scheme in [1]. In section 3.2 we derive the subdivision scheme for the control mesh of a B-spline patch with semi-open uniform knot vectors (knot vector which is open for only one side) in which, unlike from the patch with uniform knot vectors, its corner and boundary faces can be recursively subdivided using different subdivision schemes. Then we extend the scheme to the subdivision for the control mesh of an open uniform B-spline patch in section 3.3.

### 3.1 Subdivision Scheme for Uniform Quadratic B-spline Patches

The subdivision of a quadratic uniform B-spline patch with knot vector $K_u$ = [–2 –1 0 1 2 3] in both directions can be derived by the standard surface subdivision [1]. We first express this B-spline patch in matrix form

$$B_u(u,v) = UM_u GM_u^t V^t, \tag{4}$$

where

$$M_u = \frac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

is the basis matrix, $G = [P_{ij}]$, $i, j = 1, 2, 3$, is the control array, $U = [u^2\ u\ 1]$ and $V = [v^2\ v\ 1]$.

Consider the sub-patch of $B_u(u, v)$ corresponding to $0 \le u \le \frac{1}{2}, 0 \le v \le \frac{1}{2}$, denoted by $B_{00}(u_1, v_1)$, where $u_1 = 2u$ and $v_1 = 2v$. $B_{00}(u_1, v_1)$ can be rewritten as

$$B_{00}(u_1, v_1) = B_u(\tfrac{u_1}{2}, \tfrac{v_1}{2}) = U_1 SM_u GM_u^t S^t V_1^t, \tag{5}$$

where

$$S = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

$U_1 = [u_1^2\ \ u_1\ \ 1]$, and $V_1 = [v_1^2\ \ v_1\ \ 1]$. $B_{00}$ must still be a quadratic uniform B-spline patch with a new control array $G^{00}$ and therefore can be represented as

$$B_{00}(u,v) = UM_u G^{00} M_u^t V^t \tag{7}$$

for some control array $G^{00}$. Comparing Eq. (7) to Eq. (5), we obtain

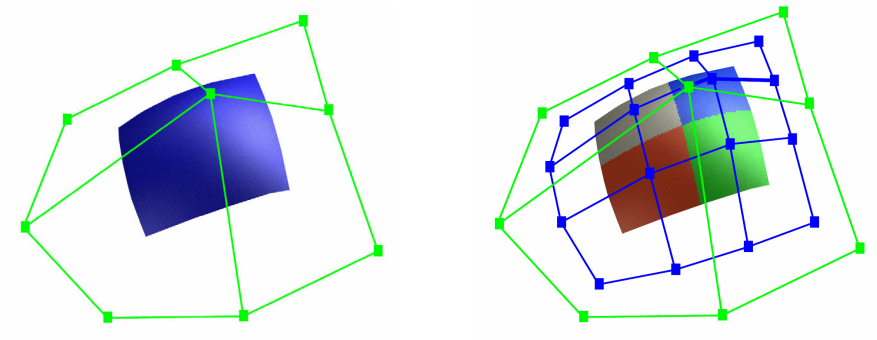
$$M_u G^{00} M_u^t = SM_u GM_u^t S^t. \tag{8}$$

Since the basis matrix $M_u$ is invertible, we have

$$\begin{aligned} G^{00} &= [M_u^{-1} SM_u] G [M_u^t SM_u^{-t}] \\ &= \frac{1}{16}\begin{bmatrix} 9P_{11}+3P_{21}+3P_{12}+P_{22} & 3P_{11}+P_{21}+9P_{12}+3P_{22} & 9P_{12}+3P_{22}+3P_{13}+P_{23} \\ 3P_{11}+9P_{21}+P_{12}+3P_{22} & P_{11}+3P_{21}+3P_{12}+9P_{22} & 3P_{12}+9P_{22}+P_{13}+3P_{23} \\ 9P_{21}+3P_{31}+3P_{22}+P_{32} & 3P_{21}+P_{31}+9P_{22}+3P_{32} & 9P_{22}+3P_{32}+3P_{23}+P_{33} \end{bmatrix}. \end{aligned} \tag{9}$$

Due to the symmetry of the B-spline basis, the other three sub-patches can be obtained similarly. The four subdivided sub-patches are still quadratic uniform B-spline patches, with 3 × 3 control arrays. However, since the control arrays of adjacent

sub-patches share the same boundary faces, a $4 \times 4$ array suffices to represent these four control meshes after the first subdivision; see Fig. 3. The subdivision scheme in general generates a new control array of dimension $2^{i-1} + 2$ at the $i^{th}$ step.



(a) Original control mesh. (b) Control mesh after one subdivision.

Fig. 3. Subdivision of a uniform quadratic B-spline patch.

### 3.2 Subdivision Scheme for Uniform Quadratic B-spline Patches With Semi-Open Knots

For the B-spline patch $B_s(u, v)$ with *semi-open* knot vector $K_s$ = [0 0 0 1 2 3] in both directions, the patch $B_s(u, v)$ can be represented in matrix form as

$$B_s(u,v) = UM_sGM_s^tV^t,$$

where

$$M_s = \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -4 & 4 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

is the basis matrix and $G$ is the control array. Using surface subdivision, $B_s$ can be split into four sub-patches $B_{00}$, $B_{01}$, $B_{10}$, and $B_{11}$ as follows:

1. $B_{00}$, corresponding to the parameter domain $0 \le u \le \frac{1}{2}, 0 \le v \le \frac{1}{2}$, is represented as a B-spline patch with the knot vector $K_s$ in both directions.
2. $B_{01}$, corresponding to the parameter domain $0 \le u \le \frac{1}{2}$, $\frac{1}{2} \le v \le 1$, is represented as a B-spline patch with knot vector $K_u$ in the $u$ direction, and $K_s$ in the $v$ direction, where $Ku$ = [–2 –1 0 1 2 3].
3. $B_{10}$, corresponding to the parameter domain $\frac{1}{2} \le u \le 1, 0 \le v \le \frac{1}{2}$, is represented as a B-spline patch with knot vector $K_s$ in the $u$ direction, and $K_u$ in the $v$ direction.
4. $B_{11}$, corresponding to the parameter domain $\frac{1}{2} \le u \le 1$, $\frac{1}{2} \le v \le 1$, is represented as a B-spline patch with knot vector $K_u$ in both directions.

$B_{00}(u, v)$ can be represented using Eq. (7), but having $M_u$ replaced by $M_s$ and using a new

$$G^{00} = [M_s^{-1} S M_s] G [M_s^t S^t M_s^{-t}] = \frac{1}{16}\begin{bmatrix} 16P_{11} & 8P_{11}+8P_{12} & 12P_{12}+4P_{13} \\ 8P_{11}+8P_{21} & 4P_{11}+4P_{21}+4P_{12}+4P_{22} & 6P_{12}+6P_{22}+2P_{13}+2P_{23} \\ 12P_{21}+4P_{31} & 6P_{21}+2P_{31}+6P_{22}+2P_{32} & 9P_{22}+3P_{32}+3P_{23}+P_{33} \end{bmatrix}. \tag{10}$$

We now show that the sub-patch $B_{11}$ can be represented by a B-spline patch with uniform knots. Since $B_{11}$ is defined in the domain $\frac{1}{2} \le u \le 1$, $\frac{1}{2} \le v \le 1$, it is $B_{11}(u_1, v_1)$, where $u_1 = 2u - 1$, $v_1 = 2v - 1$, and can be written as

$$B_{11}(u_1, v_1) = B_s(\tfrac{u_1+1}{2}, \tfrac{v_1+1}{2}) = U_1 S_1 M_s G M_s^t S_1^t V_1^t, \tag{11}$$

where

$$S_1 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}. \tag{12}$$

$B_{11}$ can also be expressed as

$$B_{11}(u, v) = U M_s G^{11} M_s^t V^t. \tag{13}$$

for some $G^{11}$. Hence the control array $G^{11}$ can be obtained as

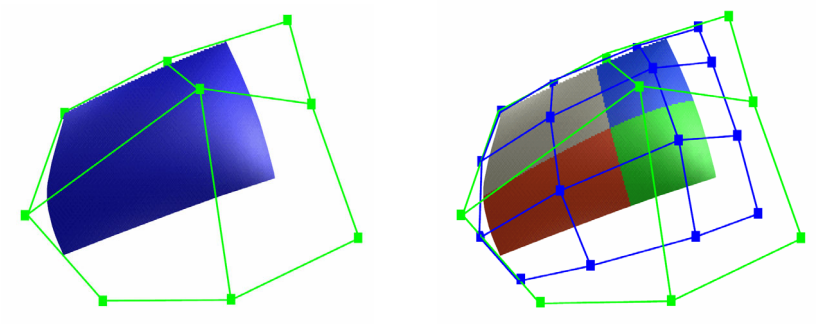
$$G^{11} = [M_u^{-1} S_1 M_s] G [M_s^t S_1^t M_u^{-t}] = \frac{1}{16}\begin{bmatrix} 4(P_{11}+P_{21}+P_{12}+P_{22}) & 6P_{12}+6P_{22}+2P_{13}+2P_{23} & 2P_{12}+2P_{22}+6P_{13}+6P_{23} \\ 6P_{21}+2P_{31}+6P_{22}+2P_{32} & 9P_{22}+3P_{32}+3P_{23}+P_{33} & 3P_{22}+P_{32}+9P_{23}+3P_{33} \\ 2P_{21}+6P_{31}+2P_{22}+6P_{32} & 3P_{22}+9P_{32}+P_{23}+3P_{33} & P_{22}+3P_{32}+3P_{23}+9P_{33} \end{bmatrix}. \tag{14}$$

The control arrays for $B_{01}$ and $B_{10}$ can be derived similarly as follows:

$$G^{01} = [M_u^{-1} S M_s] G [M_s^t S_1^t M_s^{-t}] = \frac{1}{16}\begin{bmatrix} 8(P_{11}+P_{12}) & 12P_{12}+4P_{13} & 4P_{12}+12P_{13} \\ 4(P_{11}+P_{21}+P_{12}+P_{22}) & 6P_{12}+6P_{22}+2P_{13}+2P_{23} & 2P_{12}+2P_{22}+6P_{13}+6P_{23} \\ 6P_{21}+2P_{31}+6P_{22}+2P_{32} & 9P_{22}+3P_{32}+3P_{23}+P_{33} & 3P_{22}+P_{32}+9P_{23}+3P_{33} \end{bmatrix}, \tag{15}$$

$$G^{10} = [M_s^{-1} S_1 M_s] G [M_s^t S^t M_u^{-t}] = \frac{1}{16}\begin{bmatrix} 8P_{11}+8P_{21} & 4P_{11}+4P_{21}+4P_{12}+4P_{22} & 6P_{12}+6P_{22}+2P_{13}+2P_{23} \\ 12P_{21}+4P_{31} & 6P_{21}+2P_{31}+6P_{22}+2P_{32} & 2P_{12}+2P_{22}+6P_{13}+6P_{23} \\ 4P_{21}+12P_{31} & 2P_{21}+6P_{31}+2P_{22}+6P_{32} & 3P_{22}+9P_{32}+P_{23}+3P_{33} \end{bmatrix}. \tag{16}$$

Note that adjacent control meshes share the same boundary faces, and thus the resulting sub-patches are smoothly joined with $C^0$ and $C^1$-continuity; see Fig. 4.



(a) Original control mesh. (b) Control mesh after one subdivision.

Fig. 4. Subdivision of a uniform quadratic B-spline patch with semi-open knots.

### 3.3 Subdivision Scheme for Open Uniform B-spline Patches

In this subsection we extend the subdivision scheme for the control mesh of a B-spline patch with semi-open knots to an open uniform B-spline patch. For an open uniform B-spline surface $B_o(u, v)$ defined by the knot vector

$$K_o^{n+4} = \left[-\frac{n}{2} \quad -\frac{n}{2} \quad -\frac{n}{2} \quad \left(-\frac{n}{2}+1\right) \quad \ldots \quad 0 \quad 1 \quad \ldots \quad \left(\frac{n}{2}-1\right) \quad \frac{n}{2} \quad \frac{n}{2} \quad \frac{n}{2}\right], \tag{17}$$

and an $(n + 1) \times (n + 1)$ control array $G$, the subdivision scheme described previously suggests the application of different subdivision schemes to the corner face, boundary face, and interior face.

#### 3.3.1 The subdivision scheme

We first observe that in the subdivision for the control mesh of a quadratic uniform B-spline patch, we obtain $N$ new vertices for an $N$-sided face. Furthermore, in the subdivision for control mesh of a uniform quadratic B-spline with semi-open knots, we obtain different subdivision schemes for the corner, boundary, and interior face. Namely, $G^{00}$ is applied to the corner face, $G^{01}$ and $G^{10}$ to the boundary face, and $G^{11}$ to the interior face. Let $F$ be a face of the control mesh for $B_o$ and be defined by four vertices, $P_{ij}$, $P_{i(j+1)}$, $P_{(i+1)j}$, and $P_{(i+1)(j+1)}$. Extending the previous scheme, we have the following three types of subdivision:

1. Suppose $F$ is a corner face in which $P_{ij}$ is the corner vertex, $P_{(i+1)j}$ and $P_{i(j+1)}$ are two boundary vertices, and $P_{(i+1)(j+1)}$ is the interior vertex. The four new vertices after the subdivision are derived using the upper-left $2 \times 2$ sub-array of $G^{00}$ as follows:

$$\begin{aligned} P'_{ij} &= P_{ij}, \\ P'_{(i+1)j} &= \frac{P_{ij}+P_{(i+1)j}}{2}, \\ P'_{i(j+1)} &= \frac{P_{ij}+P_{i(j+1)}}{2}, \\ P'_{(i+1)(j+1)} &= \frac{P_{ij}+P_{(i+1)j}+P_{i(j+1)}+P_{(i+1)(j+1)}}{4}. \end{aligned} \tag{18}$$

2. Suppose $F$ is a boundary face in which $P_{ij}$ and $P_{(i+1)j}$ are the boundary vertices, and $P_{i(j+1)}$ and $P_{(i+1)(j+1)}$ are the interior vertices. The four new vertices after the subdivision are derived using the upper-right $2 \times 2$ sub-array of $G^{01}$ or the lower-left $2 \times 2$ sub-array of $G^{10}$ as follows:

$$\begin{aligned} P'_{ij} &= \frac{3P_{ij}+P_{(i+1)j}}{4}, \\ P'_{(i+1)j} &= \frac{P_{ij}+3P_{(i+1)j}}{4}, \\ P'_{(i+1)(j+1)} &= \frac{6P_{ij}+2P_{(i+1)j}+2P_{i(j+1)}+6P_{(i+1)(j+1)}}{16}, \\ P'_{i(j+1)} &= \frac{2P_{ij}+6P_{(i+1)j}+6P_{i(j+1)}+2P_{(i+1)(j+1)}}{16}. \end{aligned} \tag{19}$$

3. Suppose $F$ is an interior face. The four new vertices after the subdivision are derived using the lower-right $2 \times 2$ sub-array of $G^{11}$ as follows:

$$\begin{aligned} P'_{ij} &= \frac{9P_{ij}+3P_{(i+1)j}+P_{i(j+1)}+3P_{(i+1)(J+1)}}{16}, \\ P'_{(i+1)j} &= \frac{3P_{ij}+9P_{(i+1)j}+3P_{i(j+1)}+P_{(i+1)(j+1)}}{16}, \\ P'_{(i+1)(j+1)} &= \frac{3P_{ij}+P_{(i+1)j}+3P_{i(j+1)}+9P_{(i+1)(j+1)}}{16}, \\ P'_{i(j+1)} &= \frac{P_{ij}+3P_{(i+1)j}+9P_{i(j+1)}+3P_{(i+1)(j+1)}}{16}. \end{aligned} \tag{20}$$

Note that it turns out that the subdivision for interior faces is the same as for the quadratic subdivision scheme in [1, 2].

### 3.3.2 Boundary control

Let the set of corner and boundary vertices along a boundary of the control mesh for the open uniform B-spline patch $B_o$ be $E = \{P_i \mid i = 1, \ldots, n + 1\}$. Also let $C(u)$ be the boundary curve defined by $E$ and knot vectors $K_o^{n+4}$ of Eq. (17). After applying one step of the quadratic subdivision scheme described by Eqs. (18) and (19), we obtain the set of new corner and boundary vertices $E' = \{P'_i \mid i = 1, \ldots, 2n\}$, where

$$\begin{aligned} P'_1 &= P_1 \\ P'_2 &= \frac{P_1+P_2}{2} \\ P'_i &= \begin{cases} \frac{3P_{i/2}+P_{(i+2)/2}}{4} & \text{for even } i,\ i = 4, \ldots, 4n-4 \\ \frac{P_{(i+1)/2}+3P_{(i+3)/2}}{4} & \text{for odd } i,\ i = 4, \ldots, 4n-4 \end{cases} \\ P'_{2n-1} &= \frac{P_{n+1}+P_n}{2} \\ P'_{2n} &= P_{n+1}. \end{aligned} \tag{21}$$

If the deBoor or Oslo knot refinement algorithm is applied to $C(u)$ by inserting uniform $(n - 1)$ knots into $K_o^{n+4}$, the resulting set of new vertices is exactly the same as $E'$.

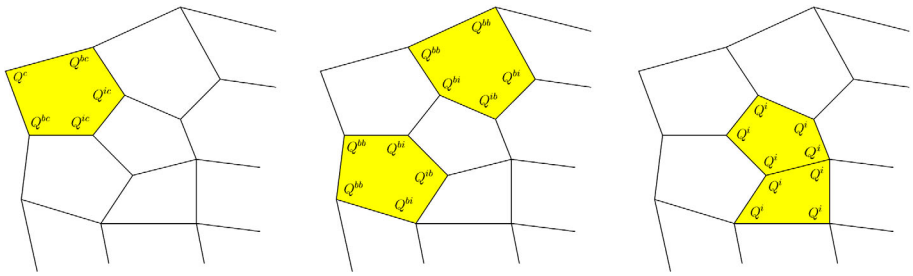
That is, the boundary curve $C(u)$ can also be defined by the new control vertices $E'$ with open uniform knot vector $K_o^{2n+3}$. Hence, we conclude that new corner and boundary vertices resulting from each subdivision step always define the boundary curve represented by the original mesh.

## 4. OPEN UNIFORM QUADRATIC SUBDIVISION SCHEME FOR CONTROL MESHES OF RESTRICTED TOPOLOGY

We consider here only control meshes with *restricted topology* [9]; this is, every boundary face contains only one boundary edge, every corner face has only two boundary edges, and furthermore, no 3-sided face is allowed. We first extend the described previously open uniform quadratic subdivision scheme to the *n*-sided corner and boundary faces, aiming to achieve the boundary control at each subdivision step. However, the cross-tangent of the subdivision surface along the boundary curves is in general not well defined when the corner and boundary faces are not all 4-sided. Later in this section, we resolve this problem by first splitting all the faces of the control mesh into 4-sided faces, then applying the open uniform quadratic subdivision scheme to the new control mesh. Consequently, both the $C^0$ and $C^1$ boundary conditions can be evaluated. This property allows us to join two subdivision surfaces with $C^0$ and $C^1$-continuity.

### 4.1 General Open Uniform Quadratic Subdivision Scheme

Note that in the subdivision of a quadratic open uniform B-spline patch, the corner face, boundary face, and interior face employ different subdivision schemes. Moreover, vertices of the corner and boundary face are assigned with different subdivision weights. In order to generalize this subdivision scheme to a control mesh with n-sided faces, we first classify the vertices of a face into the following types (see Fig. 5):




(a) Corner face. (b) Boundary face. (c) Interior face.

Fig. 5. General quadratic open uniform subdivision scheme.

- For a corner face, we have the following types of vertices:
  - $Q^c$ - the corner vertex
  - $Q^{bc}$ - a boundary vertex adjacent to a vertex of type $Q^c$
  - $Q^{ic}$ - an interior vertex

- For a boundary face, we have the following types of vertices:
  - $Q^{bb}$ - a boundary vertex
  - $Q^{bi}$ - an interior vertex adjacent to a vertex of type $Q^{bb}$
  - $Q^{ib}$ - an interior vertex that is not adjacent to a vertex of type $Q^{bb}$

- For an interior face, we have only one type of vertex, $Q^i$, which is the interior vertex of the interior face.

Let $F_k$ be an $n$-sided face defined by vertices $Q_j$, $j = 1, \ldots, n$. After one step of the subdivision, we obtain a new control vertex $Q'_i$ for each $Q_i$ by using

$$Q_i^{'} = \sum_{j=1}^{n} \alpha_{ij} Q_j ,$$

where the weights $\alpha_{ij}$ are computed based on the classification of $Q_i$ according to the following rules:

- For a corner face, the weights $\alpha_{ij}$ are derived as follows:
  - Suppose $Q_i$ is a corner vertex of type $Q^c$ (see Fig. 5 (a)). As in Eq. (18) the new corner vertex $Q'_i$ coincides with the old corner vertex $Q_i$. This implies that the weights are all zero except the one for $Q_i$; that is,

$$\alpha_{ij} = \begin{cases} 1 & j = i, \\ 0 & j \neq i. \end{cases}$$

  - Suppose $Q_i$ is a boundary vertex of type $Q^{bc}$ in the corner face, and $Q_k$, an adjacent vertex of $Q_i$, is of type $Q^c$ (see Fig. 5 (a)). As in Eq. (18), the new boundary vertex $Q'_i$ is the average of the original boundary vertices and the adjacent corner vertex $Q_k$. Hence we have

$$\alpha_{ij} = \begin{cases} \frac{1}{2} & j = i, \\ \frac{1}{2} & j = k, \\ 0 & j \neq i,\ j \neq k. \end{cases}$$

  - Suppose $Q_i$ is an interior vertex of type $Q^{ic}$ in the corner face (see Fig. 5 (a)). The $P'_{(i+1)(j+1)}$ in Eq. (18) indicates that the new vertex of an interior vertex of the 4-sided corner face is the centroid of the face, i.e., all old vertices are uniformly weighted. However, for faces with more than four sides, this implies that all the new interior vertices will be coincident. To circumvent such an result, the weight of $Q_i$ is set to $(n - 3)/n$, and all other old vertices share the remaining weight $1 - (n - 3)/n$, that is,

$$\alpha_{ij} = \begin{cases} \frac{(n-3)}{n} & j = i, \\ \frac{3}{n(n-1)} & j \neq i. \end{cases}$$

- For a boundary face, we use the following weights:
  - Suppose $Q_i$ is a boundary vertex of type $Q^{bb}$, and $Q_k$ is a vertex adjacent to $Q_i$ and of type $Q^{bb}$ (see Fig. 5 (b)). As is $P'_{ij}$ in Eq. (19), the new boundary vertex $P'_{ij}$ is the weighted average of the two adjacent boundary vertices $P_{ij}$ and $P_{(i+1)j}$. Hence we have

$$\alpha_{ij} = \begin{cases} \frac{3}{4} & j = i, \\ \frac{1}{4} & j = k, \\ 0 & j \neq i, j \neq k. \end{cases}$$

  - Suppose $Q_i$ is an interior vertex of type $Q^{bi}$, and $Q_k$ is a vertex adjacent to $Q_i$ of type $Q^{bb}$ (see Fig. 5 (b)). As ix $P'_{(i+1)(j+1)}$ in Eq. (19), the two vertices $P_{(i+1)(j+1)}$ and $P_{ij}$ share the weight 3/4, and all other vertices share the weight $1 - 3/4$. The weights for a $n$-sided boundary face here can be obtained by replacing the denominator 4 by $n$, resulting in the following weight assignment:

$$\alpha_{ij} = \begin{cases} \frac{3}{2n} & j = i, \\ \frac{3}{2n} & j = k, \\ \frac{n-3}{n(n-2)} & j \neq i, j \neq k. \end{cases}$$

  - Suppose $Q_i$ is an interior vertex of type $Q^{ib}$ in a boundary face (see Fig. 5 (b)). Since there is no corresponding formulation from Eq. (19), here we use the weights as in the case of corner vertex $Q^{ic}$ for a corner face:

$$\alpha_{ij} = \begin{cases} \frac{(n-3)}{n} & j = i, \\ \frac{3}{n(n-1)} & j \neq i, \end{cases}$$

- For the vertices $Q_i$ of type $Q^i$ in an interior face (see Fig. 5 (c)), we apply Eq. (20) and obtain the same weight used in the quadratic subdivision schemes, i.e., the weight shown in Eq. (2) or (3).

We term the above subdivision scheme the *open uniform quadratic subdivision*, and the resulting limit surface the *open uniform quadratic subdivision surface*.

### 4.2 Boundary Control and Surface Properties

Since the open uniform quadratic subdivision scheme for the control mesh of restricted topology is a generalization of the subdivision scheme used for rectangular open uniform B-spline patches, and in both schemes the corner and boundary vertices are derived using the same weights shown in Eq. (21), each boundary curve of the subdivision surface is an open uniform B-spline curve defined by the corner and boundary vertices at each subdivision step, and, consequently, the boundary curve of the open uniform quadratic subdivision surface is an open uniform B-spline curve defined by the original mesh.

Since the new vertices are derived using the affined combination of old face vertices and the weights sum to one, each boundary face must be smaller than its parent face after one step of the subdivision. Hence, all boundary faces of the control mesh converge to the boundary curves of the open uniform quadratic subdivision surface. Since the interior faces at each subdivision step are subdivided using the standard quadratic subdivision scheme, as the boundary faces converge to the boundary curves of the subdivision surface, the continuity conditions in the interior of the open uniform quadratic subdivision surface are similar to the subdivision surfaces derived using the standard quadratic subdivision scheme; see [2, 3, 5, 6].

We conclude that the open uniform quadratic subdivision surface is $C^1$-continuous in its interior. And the boundary curves of the open uniform quadratic surface are the open uniform B-spline curves defined by the boundary vertices at each subdivision step, including the original mesh.

### 4.3 Cross-Tangent Derivation

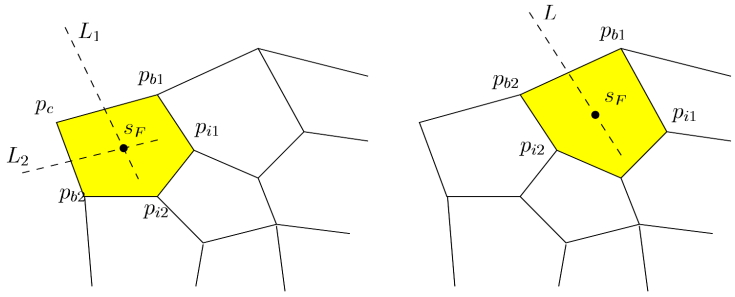
Although the open uniform quadratic subdivision scheme achieves boundary control, its cross-tangent condition is in general hard to evaluate. For a control mesh with 4-sided corner and boundary faces, it is possible to derive the cross-tangent along the boundary curve. Given a general control mesh with restricted topology, in order to derive the cross-tangent along a boundary curve we first split each n-sided face into n 4-sided faces by connecting the *split vertex* $s_F$ of the face to the middle point on each face's edge. The split vertex is chosen such that the $C^1$-continuity between two open uniform quadratic subdivision surfaces can be achieved. Rules for selecting the split vertex are as follows:

- For a corner face, suppose that $p_c$ is the corner vertex, $p_{b1}$ and $p_{b2}$ are the two boundary vertices adjacent to $p_c$, and $p_{i1}$ and $p_{i2}$ are the interior vertices adjacent to $p_{b1}$ and $p_{b2}$, respectively. Let $L_1$ be the bisection line of $\overline{p_c p_{b2}}$ and $\overline{p_{b1} p_{i1}}$ and $L_2$ be the bisection line of $\overline{p_c p_{b1}}$ and $\overline{p_{b2} p_{i2}}$. The split vertex $s_F$ is chosen as the intersection of $L_1$ and $L_2$; see Fig. 6 (a).

- For a boundary face, suppose that $p_{b1}$ and $p_{b2}$ are the two boundary vertices, and $p_{i1}$ and $p_{i2}$ are the interior vertices adjacent to $p_{b1}$ and $p_{b2}$, respectively. Let $L$ be the bisection line of $\overline{p_{b1} p_{i1}}$ and $\overline{p_{b2} p_{i2}}$. The split vertex is the projection of the face's centroid onto the line $L$, see Fig. 6 (b).

- For an interior face, the centroid of the face is used as the split vertex.

The face splitting results in a new control mesh having only four-sided faces. The subdivision scheme described in section 4.1 can be directly applied to the new control mesh. As described in previous sections, the resulting subdivision surface is also $C^1$-continuous inside the surface.




(a) Corner face. (b) Boundary face.

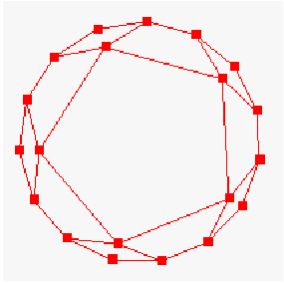
Fig. 6. The selection of split vertex for corner and boundary faces.

We now show that two subdivision surfaces of this kind can be joined with $C^1$-continuity. Let $S^1$ and $S^2$ be two open uniform quadratic subdivision surfaces derived from control meshes with restricted topology, $M^1$ and $M^2$, respectively. And let $p_1$, $p_2$, …, and $p_m$ be the boundary vertices along a common boundary of $M^1$ and $M^2$. There must be $m$ interior vertices $q_1^1$, $q_2^1$, ..., and $q_m^1$ of $M^1$, in which each $q_i^1$ is adjacent to $p_i$, for $i = 1, 2, \ldots, m$. Similarly, there must be $m$ interior vertices $q_1^2$, $q_2^2$, ..., and $q_m^2$ of $M^2$, in which each $q_i^2$ is adjacent to $p_i$, for $i = 1, 2, \ldots, m$. We refer to each set $\{p_i, q_i^1, q_i^2\}$, for $i = 1, 2, \ldots, m$, as the *cross-tangent triple* of control meshes $M^1$ and $M^2$. After applying the face splitting to $M^1$ and $M^2$, we obtain control meshes $M^{1'}$ and $M^{2'}$ with only 4-sided faces and also obtain new cross-tangent triples $\{p_i', q_i^{1'}, q_i^{2'}\}$, $i = 1, \ldots, 2m-1$, between $M^{1'}$ and $M^{2'}$. If all the triples $\{p_i, q_i^1, q_i^2\}$, $i = 1, \ldots, m$, for the original control meshes are co-linear, the proposed face splitting scheme ensures that all the new triples $\{p_i', q_i^{1'}, q_i^{2'}\}$, are also co-linear. Thus two open uniform quadratic subdivision surfaces derived from $M^{1'}$ and $M^{2'}$ are joined with $C^0$ and $C^1$-continuity at every step of the subdivision.

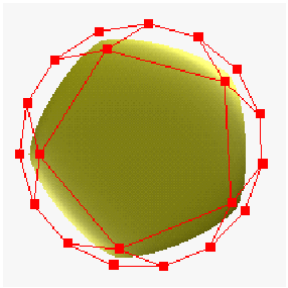
## 5. EXAMPLES

Fig. 7 shows an original control mesh, its shaded subdivision surface using the standard quadratic subdivision, and its shaded open uniform quadratic subdivision surface. Fig. 8 shows the result of a 6-sided control mesh with concave edges. All subdivision surfaces are the results from 6 steps of the subdivision. Experimental results show that the proposed scheme can archive effective boundary control for quadratic subdivision surfaces.

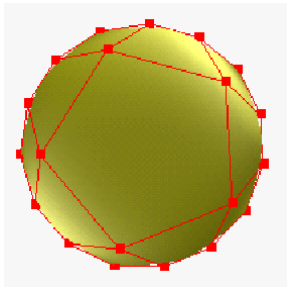
Fig. 9 (b) shows the splitting of the original control mesh shown in Figs. 9 (a) and (c) show the resulting open uniform quadratic subdivision surface. Figs. 10 (a) and (b) show the join of two open uniform quadratic subdivision surfaces with $C^0$-continuity and $C^1$-continuity, respectively.



(a) Original mesh.

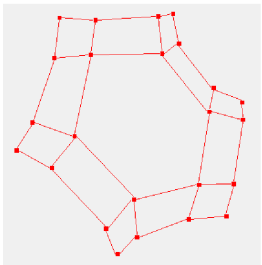


(b) Subdivision surface using standard quadratic scheme (together with the original mesh).

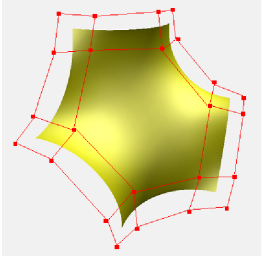


(c) Subdivision surface using open uniform quadratic subdivision scheme (together with the original mesh).

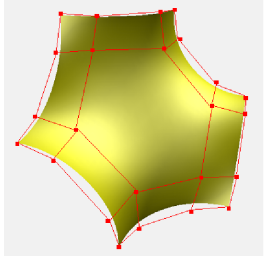
Fig. 7. Boundary control of 5-sided control mesh using open uniform quadratic subdivision scheme.



(a) Original mesh.

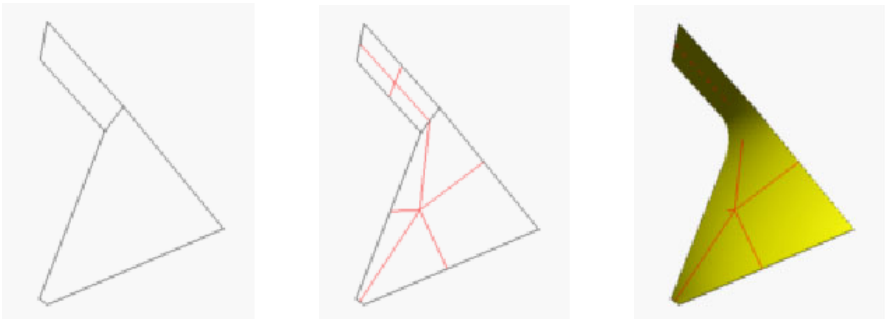


(b) Subdivision surface using standard quadratic scheme (together with the original mesh).



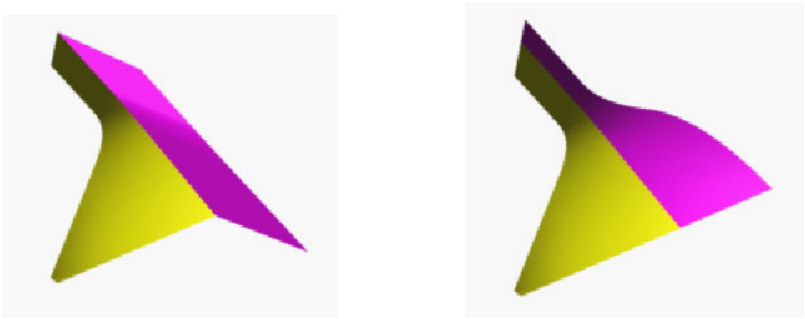
(c) Subdivision surface using open uniform quadratic subdivision scheme (together with the original mesh).

Fig. 8. Boundary control of 6-sided control mesh using open uniform quadratic subdivision scheme.



(a) Original mesh. (b) Splitting of faces. (c) Subdivision surface.

Fig. 9. The open uniform quadratic subdivision surface by face splitting.



(a) With $C^0$-continuity. (b) With $C^1$-continuity.

Fig. 10. Joining two open uniform quadratic subdivision surfaces.

## 6. CONCLUDING REMARKS

The boundary control of $C^0$ and $C^1$-continuity is one of several features that the subdivision surface must have before it becomes a useful representation in computer aided geometric design. We have proposed an open uniform quadratic subdivision scheme that is derived from the subdivision of the open uniform B-spline surfaces. In the proposed scheme, the open uniform quadratic subdivision is applied to all corner and boundary faces and a standard quadratic subdivision is applied to all interior faces of a given mesh with restricted topology. We have also shown that for control meshes with 4-sided corner and boundary faces, the cross-tangent condition can be defined during every step of the open uniform quadratic subdivision, and hence the subdivision surfaces derived from two control meshes of this type can be joined with $C^0$ and $C^1$-continuity. In another work, we have applied the proposed open uniform quadratic subdivision surface to fill an N-sided hole on a parametric surface and to derive the vertex blend of parametric surfaces.

## ACKNOWLEDGEMENT

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