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ON DISCRETE INEQUALITIES OF GRÜSS TYPE

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Abstract. In the paper, we establish two discrete inequalities of Grüss type via inequalities of Watson-Greub-Rheiuboldt and Klamkin-Mclenaghan.

1. Introduction

In [6], Grüss proved the following integral inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \le \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma) \quad (1.1)$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the condition

$$\phi \le f(x) \le \Phi \text{ and } \gamma \le g(x) \le \Gamma \text{ for } a.e.x \in [a,b]. \quad (1.2)$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The discrete version of (1.1) states that (see for example [2]):

If $\overline{a} = (a_1, \dots a_n)$ and $\overline{b} = (b_1, \dots b_n)$ are two sequences of real numbers with

$$a \le a_i \le A < \infty \text{ and } b \le b_i \le B < \infty \text{ for all } i \in \{1, \dots, n\},$$

then

$$\begin{aligned} \left|C_n(\overline{a}, \overline{b})\right| &\le \frac{1}{n}[\frac{1}{2}](1-\frac{1}{n}[\frac{1}{2}])(A-a)(B-b) \\ &= \frac{1}{n^2}[\frac{n^2}{4}](A-a)(B-b) \\ &\le \frac{1}{4}(A-a)(B-b) \quad (1.3) \end{aligned}$$

where

$$C_n(\overline{a}, \overline{b}) = \frac{1}{n}\sum_{i=1}^n a_ib_i - \frac{1}{n}\sum_{i=1}^n a_i \cdot \frac{1}{n}\sum_{i=1}^n b_i.$$

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For other new results in the domain, see the papers [1, 3-8, 9-10] and the book [8].

Recently, Dragomir and Khan [5] proved the following two discrete inequalities of Grüss type:

Theorem A. *Let $\overline{a} = (a_1, \ldots a_n)$ and $\overline{b} = (b_1, \ldots b_n)$ be two n-tuples of positive real numbers with*

$$0 < a \leq a_i \leq A < \infty \text{ and } 0 < b \leq b_i \leq B < \infty \text{ for all } i \in \{1, \ldots, n\}.$$

Then we have the inequality

$$\left|C_n(\overline{a}, \overline{b})\right| \leq \frac{1}{4} \frac{(A-a)(B-b)}{\sqrt{AaBb}} \cdot \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i. \tag{1.4}$$

The constant $\frac{1}{4}$ is the best possible.

Theorem B. *Let $\overline{a} = (a_1, \ldots a_n)$ and $\overline{b} = (b_1, \ldots b_n)$ be defined as in Theorem A. Then we have the inequality*

$$\left|C_n(\overline{a}, \overline{b})\right| \leq (\sqrt{A} - \sqrt{a})(\sqrt{B} - \sqrt{b}) \cdot \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i. \tag{1.5}$$

The constant $c = 1$ is the best possible.

In this paper, we establish a weighted generalization of Theorem A and Theorem B, respectively.

2. Main Results

Theorem 1. *Let $\overline{a} = (a_1, \ldots a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ be defined as in Theorem A, and let $\overline{p} = (p_1, \ldots, p_n)$ be an n-tuples of nonnegative numbers with $P_n = \sum_{i=1}^{n} p_i > 0$. Then we have the inequality*

$$\left|C_n(\overline{p}, \overline{a}, \overline{b})\right| \leq \frac{1}{4} \frac{(A-a)(B-b)}{\sqrt{AaBb}} \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \tag{2.1}$$

where

$$C_n(\overline{p}, \overline{a}, \overline{b}) = \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i$$

and the constant $\frac{1}{4}$ is the best possible.

Proof. By the Cauchy-Schwarz inequality, we have

$$\left|C_n(\overline{p}, \overline{a}, \overline{b})\right| = \left|\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i\right|$$

$$= \left| \frac{1}{2P_n^2} \sum_{i,j=1}^{n} p_i p_j (a_i - a_j)(b_i - b_j) \right|$$

$$\leq \frac{1}{2P_n^2} \sum_{i,j=1}^{n} p_i p_j \left|(a_i - a_j)\right| \left|(b_i - b_j)\right|$$

$$\leq \frac{1}{2P_n^2} [\sum_{i,j=1}^{n} p_i p_j (a_i - a_j)^2]^{\frac{1}{2}} [\sum_{i,j=1}^{n} p_i p_j (b_i - b_j)^2]^{\frac{1}{2}}$$

$$= \frac{1}{2P_n^2} [2P_n \sum_{i=1}^{n} p_i a_i^2 - 2(\sum_{i=1}^{n} p_i a_i)^2]^{\frac{1}{2}} [2P_n \sum_{i=1}^{n} p_i b_i^2 - 2(\sum_{i=1}^{n} p_i b_i)^2]^{\frac{1}{2}}$$

$$= [\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i^2 - (\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i)^2]^{\frac{1}{2}} [\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i^2 - (\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i)^2]^{\frac{1}{2}}. \tag{2.2}$$

Using the Waston-Greub-Rheiuboldt inequality [8, p.122]

$$\sum_{i=1}^{n} w_i z_i^2 \cdot \sum_{i=1}^{n} w_i u_i^2 \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} \cdot (\sum_{i=1}^{n} w_i z_i u_i)^2, \tag{2.3}$$

provided $0 \leq w_i < \infty$, $0 < m_1 \leq z_i \leq M_1 < \infty$ and $0 < m_2 \leq u_i \leq M_2 < \infty$ $(i = 1, \ldots, n)$, we have

$$(\sum_{i=1}^{n} p_i a_i^2) \cdot P_n \leq \frac{(A + a)^2}{4aA} \cdot (\sum_{i=1}^{n} p_i a_i)^2$$

giving

$$\frac{(\sum_{i=1}^{n} p_i a_i^2) \cdot P_n - (\sum_{i=1}^{n} p_i a_i)^2}{(\sum_{i=1}^{n} p_i a_i)^2} \leq \frac{(A + a)^2 - 4aA}{4aA} = \frac{(A - a)^2}{4aA},$$

thas is,

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i^2 - \frac{1}{P_n} (\sum_{i=1}^{n} p_i a_i)^2 \leq \frac{(A - a)^2}{4aA} \cdot (\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i)^2. \tag{2.4}$$

Similarly, we have

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i^2 - \frac{1}{P_n} (\sum_{i=1}^{n} p_i b_i)^2 \leq \frac{(B - b)^2}{4bB} \cdot (\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i)^2. \tag{2.5}$$

Using (2.2), (2.4) and (2.5), we otain the inequality (2.1). As note in the proof of Theorem in [5], we obtain that the constant $\frac{1}{4}$ is the best possible. This completes the proof.

Remark 1. Choose $p_i = \frac{1}{n}$ $(1,\ldots,n)$ in Theorem 1. Then the inequality (2.1) reduces to (1.4).

Theorem 2. *Let $\bar{a} = (a_1,\ldots,a_n)$, $\bar{b} = (b_1,\ldots,b_n)$ and $\bar{p} = (p_1,\ldots,p_n)$ be defined as in Theorem 1. Then we have*

$$|C_n(\bar{p},\bar{a},\bar{b})| \le (\sqrt{A}-\sqrt{a})(\sqrt{B}-\sqrt{b})\Big(\frac{1}{P_n}\sum_{i=1}^n p_i a_i\Big)^{\frac12}\Big(\frac{1}{P_n}\sum_{i=1}^n p_i b_i\Big)^{\frac12}. \tag{2.6}$$

The constant $c = 1$ is the best possible.

Proof. Using the Klamkin-Mclenaghan inequality [7, p.125]

$$\sum_{i=1}^n w_i z_i^2 \cdot \sum_{i=1}^n w_i u_i^2 - \Big(\sum_{i=1}^n w_i z_i u_i\Big)^2 \le (\sqrt{M}-\sqrt{m})^2 \sum_{i=1}^n w_i z_i u_i \cdot \sum_{i=1}^n w_i u_i^2, \tag{2.7}$$

provided $0 \le w_i < \infty$, $z_i > 0$, $u_i > 0$ and $0 < m \le \frac{z_i}{u_i} \le M < \infty$ $(i = 1,\ldots,n)$, we have

$$\Big(\sum_{i=1}^n p_i a_i^2\Big)\cdot P_n - \Big(\sum_{i=1}^n p_i a_i\Big)^2 \le (\sqrt{A}-\sqrt{a})^2\Big(\sum_{i=1}^n p_i a_i\Big)\cdot P_n$$

that is,

$$\frac{1}{P_n}\sum_{i=1}^n p_i a_i^2 - \Big(\frac{1}{P_n}\sum_{i=1}^n p_i a_i\Big)^2 \le (\sqrt{A}-\sqrt{a})^2\Big(\frac{1}{P_n}\sum_{i=1}^n p_i a_i\Big). \tag{2.8}$$

Similarly, we have

$$\frac{1}{P_n}\sum_{i=1}^n p_i b_i^2 - \Big(\frac{1}{P_n}\sum_{i=1}^n p_i b_i\Big)^2 \le (\sqrt{B}-\sqrt{b})^2\Big(\frac{1}{P_n}\sum_{i=1}^n p_i b_i\Big). \tag{2.9}$$

Using (2.2), (2,8) and (2.9), we obtain the inequality. As note in proof Theorem 1 in [5], we obtain that the constant $c = 1$ is the best possible. This completes the proof.

Remark 2. Choose $p_i = \frac{1}{n}$ $(1,\ldots,n)$ in Theorem 2. Then the inequality (2.6) reduces to (1.5).

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