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Global Stability for the Leslie-Gower Predator-Prey System with Time-Delay and Holling's Type Functional Response

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Abstract

In this paper, we concerned about the dynamical behavior of the Leslie-Gower predator-prey system with time delay and different functional response $p(x)$. At first, we discuss the global stability for the Leslie-Gower predator-prey system with time delay and $p(x) = cx$. Secondly, with $p(x) = \frac{cx}{1+x}$. Thirdly, with $p(x) = \frac{cx^2}{1+x^2}$. Finally, we illustrate our results by some examples.

1 Introduction

Predator-prey models have been studied for a long time. One of the most important problems in a predator-prey system is the global stability of the unique positive equilibrium point. The global stability analysis for predator-prey system without time delay has been done by many researchers. Most of them use the following methods to prove global stability of a predator-prey system without delay. The first method is to construct a Lyapunov function [3,4,5]. The second method is to employ the Dulac Criterion to eliminate the existence of periodic orbits and then use the Poincaré-Bendixson Theorem to analyze the global stability of the unique positive equilibrium [3,4,5,6]. The third method is the limit cycle stability analysis [3,6,7,8]. The fourth method is the comparison method [3,7,8].

But more realistic models should include some of the past states of the population system; that is, a real system should be modeled with time delays. In [9,10,11], authors were to analyze the global stability of the system with time delay by constructing a Lyapunov functional.

In this paper, we were concerned about the Leslie-Gower predator-prey system. For this system without delay as in [12], authors to analyze the global stability by constructing a Lyapunov

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function. And in [14], authors discussed the global stability of this system with a single delay by constructing a Lyapunov functional. Now, we are to establish global stability of the Leslie-Gower predator-prey system with a single delay with the different functional response of the predator $p(x)$, by constructing a Lyapunov functional. In section 2, we analyze the global stability of the Leslie-Gower predator-prey system with a single delay with $p(x) = cx$ in the Holling-type I, $p(x) = \frac{cx}{1+x}$ in the Holling-type II, and $p(x) = \frac{cx^2}{1+x^2}$ in the Holling-type III by constructing Lyapunov functionals. In section 3, we illustrate our results by some examples.

2 The Model with Time Delay

Consider the Leslie-Gower predator-prey system with time delay τ modeled by

$$\begin{aligned} \dot{x}(t) &= x(t)\left\{r\left[1-\frac{x(t-\tau)}{K}\right]\right\} - p(x(t))y(t) \\ \dot{y}(t) &= y(t)\left[\delta - \beta\frac{y(t)}{x(t)}\right] \end{aligned} \tag{2.1}$$

with the initial conditions

$$\begin{aligned} &x(\theta) = \phi(\theta) \geq 0\,,\ \theta \in [-\tau,\infty)\,.\ \phi \in C^1([-\tau,\infty),\mathbb{R}) \\ &x(0) > 0\ ,\ y(0) > 0 \end{aligned} \tag{2.2}$$

where δ, β, r, and τ are positive constants. K is defined as the prey environmental carrying capacity. x and y denote the densities of prey and predator population, respectively. Because all we want to discuss is biological population, we only consider the first quadrant in the $x-y$ plane. The following assumption is consistent with the system (2.1).

(A) $p \in C^1([0,\infty),[0,\infty))$; $p(0) = 0$ and $p'(x) \geq 0$ for all $x > 0$.

The functional response of the predator, $p(x)$, has been discussed in the literature. Now, we just concern about the following $p(x)$ of this paper, $p(x) = cx$ in the Holling-type I model, $p(x) = \frac{cx}{1+x}$ in the Holling-type II model, and $p(x) = \frac{cx^2}{1+x^2}$ in the Holling-type III model; where c is encounter rate.

Lemma 2.1 *Every solution of the system (2.1) with the initial conditions (2.2) exists in the interval* $[0,\infty)$ *and remains positive for all* $t \geq 0$.

Proof: It is true because

$$\begin{aligned} x(t) &= x(0)exp\left\{\int_0^t \left[r - \frac{rx(s-\tau)}{K} - \frac{p(x(s))}{x(s)}y(s)\right] ds\right\} \\ y(t) &= y(0)exp\left\{\int_0^t \left[\delta - \beta\frac{y(s)}{x(s)}\right] ds\right\} \end{aligned}$$

and $x(0), y(0) > 0$.

Lemma 2.2 *Let* $(x(t), y(t))$ *denote the solution of (2.1) with the initial conditions (2.2), then*

$$0 < x(t) \leq M, \; 0 < y(t) \leq L \tag{2.3}$$

eventually for all large t, where

$$M = Ke^{r\tau} \tag{2.4}$$

$$L = \frac{\delta M}{\beta} \tag{2.5}$$

Proof: Now, we want to show that there exists a $T > 0$ such that $x(t) \leq M$ for $t > T$. By Lemma 2.1, we know that solutions of the system (2.1) are positive, and hence, by assumption (A), and (2.1)

$$\begin{aligned} \dot{x}(t) &= x(t)\left\{r\left[1 - \frac{x(t-\tau)}{K}\right]\right\} - p(x(t))y(t) \\ &\leq rx(t)\left[1 - \frac{x(t-\tau)}{K}\right] \end{aligned} \tag{2.6}$$

Taking $M^* = K(1+K_1)$, $0 < K_1 < e^{r\tau} - 1$. Suppose $x(t)$ is not oscillatory about M^*. That is, there exists a $T > 0$ such that either

$$x(t) > M^* \qquad \text{for } t > T_0 \tag{2.7}$$

or

$$x(t) \leq M^* \qquad \text{for } t > T_0 \tag{2.8}$$

If (2.8) holds, then for $t > T_0$

$$x(t) \leq M^* = K(1+K_1) < Ke^{r\tau} = M$$

That is, (2.3) holds. Suppose (2.7) holds. Equation (2.6) implies that for $t > T_0 + \tau$

$$\begin{aligned} \dot{x}(t) &\leq rx(t)\left[1 - \frac{x(t-\tau)}{K}\right] \\ &< -K_1 rx(t) \end{aligned}$$

It follows that

$$\int_{T_0+\tau}^{t} \frac{\dot{x}(s)}{x(s)} ds < \int_{T_0+\tau}^{t} -K_1 r ds = -K_1 r(t - T_0 - \tau)$$

Then $0 < x(t) < x(T_0+\tau)e^{-K_1 r(t-T_0-\tau)} \to 0$ as $t \to \infty$. That is, $\lim_{t\to\infty} x(t) = 0$ by the Squeeze Theorem. It contradicts to (2.7). Therefore, there must exist a $T_1 > T_0$ such that $x(T_1) \leq M^*$. If $x(T_1) \leq M^*$ for all $t \geq T_1$, then (2.3) follows. If not, then there must exist a $T_2 > T_1$ such that T_2 be the first time which $x(T_2) > M^*$. Therefore, there exists a $T_3 > T_2$ such that T_3 be the first time which $x(T_3) < M^*$ by above discussion. By above, we know that $x(T_1) \leq M^*$, $x(T_2) > M^*$, and $x(T_3) \leq M^*$ where $T_1 < T_2 < T_3$. Then, by the Intermediate Value Theorem, there exists T_4 and T_5 such that

$$x(T_4) = M^* \quad , \quad T_1 \leq T_4 < T_2$$
$$x(T_5) = M^* \quad , \quad T_2 \leq T_5 < T_3$$

and $x(t) > M^*$ for $T_4 < t < T_5$. Hence there is a $T_6 \in (T_4, T_5)$ such that $x(T_6)$ is an arbitrary local maximum, and hence it follows from (2.6) that

$$0 = \dot{x}(T_6) \leq rx(T_6)\left[1 - \frac{x(T_6-\tau)}{K}\right]$$

and this implies

$$x(T_6-\tau) \leq K$$

Integrating both sides of (2.6) on the interval $[T_6-\tau, T_6]$, we have

$$\ln\left[\frac{x(T_6)}{x(T_6-\tau)}\right] = \int_{T_6-\tau}^{T_6} \frac{\dot{x}(s)}{x(s)} ds \leq \int_{T_6-\tau}^{T_6} r\left[1 - \frac{x(s-\tau)}{K}\right] ds \leq r\tau$$

It follows that

$$x(T_6) \leq x(T_6-\tau)e^{r\tau} \leq Ke^{r\tau} = M$$

Since $x(T_6)$ is local maximum of $x(t)$ and $x(T_6) \leq M$, $x(t) \leq M$ where t near T_6. Since $x(T_6)$ is an arbitrary local maximum of $x(t)$, we can conclude that there exists a $T > 0$ such that

$$x(t) \leq M \qquad \text{for } t \geq T \tag{2.9}$$

Suppose $x(t)$ is oscillatory about M^*, for this case, the proof is similarly to above one. Now, we want to show that $y(t)$ is bounded above by L eventually for all large t. By (2.9), it follows that for

$t > T$

$$\begin{aligned}\dot{y}(t) &= y(t)\left[\delta - \beta\frac{y(t)}{x(t)}\right] \\ &\leq y(t)\left[\delta - \frac{\beta}{M}y(t)\right] \\ &= \delta y(t)\left[1 - \frac{\beta}{\delta M}y(t)\right] \\ &= \delta y(t)\left[1 - \frac{y(t)}{\frac{\delta M}{\beta}}\right]\end{aligned}$$

Therefore, $y(t) \leq \frac{\delta M}{\beta} = L$ for $t > T$. This completes the proof.

Lemma 2.3 *Suppose that the system (2.1) satisfies*

$$r - cL > 0 \tag{2.10}$$

where L defined by (2.5), and c is defined in assumption (A). Then the system (2.1) is uniformly persistent. That is, there exists m, l, and $T^* > 0$ *such that* $m \leq x \leq M$ *and* $l \leq y \leq L$ *for* $t \geq T^*$.

Proof: By Lemma 2.2, and assumption (A), equation (2.1) follows that for $t \geq T + \tau$

$$\dot{x}(t) \geq x(t)\left[r\left(1 - \frac{M}{K}\right) - cL\right] \tag{2.11}$$

Integrating both sides of (2.11) on $[t-\tau, t]$, where $t \geq T + \tau$, then we have

$$x(t) \geq x(t-\tau)e^{[r(1-\frac{M}{K})-cL]\tau}$$

That is

$$x(t-\tau) \leq x(t)e^{-[r(1-\frac{M}{K})-cL]\tau} \tag{2.12}$$

It follows from (2.1) that for $t \geq T + \tau$

$$\begin{aligned}\dot{x}(t) &= x(t)r\left[1 - \frac{x(t-\tau)}{K}\right] - p(x(t))y(t) \\ &\geq x(t)\left\{r - cL - \frac{r}{K}e^{-[r(1-\frac{M}{K})-cL]\tau}x(t)\right\} \\ &= (r - cL)x(t)\left\{1 - \frac{x(t)}{\frac{K(r-cL)}{r}e^{[r(1-\frac{M}{K})-cL]\tau}}\right\}\end{aligned}$$

It follows that

$$\liminf_{t\to\infty} x(t) \geq \frac{K(r-cL)}{r}e^{[r(1-\frac{M}{K})-cL]\tau} \equiv \overline{m}$$

and $\overline{m} > 0$ by (2.10). So, for large t, $x(t) > \frac{\overline{m}}{2} \equiv m > 0$. It follows that

$$\begin{aligned}
\dot{y}(t) &\geq y(t)\left[\delta - \beta\frac{y(t)}{m}\right] \\
&= \delta y(t)\left[1 - \frac{\beta}{\delta m}y(t)\right] \\
&= \delta y(t)\left[1 - \frac{y(t)}{\frac{\delta m}{\beta}}\right]
\end{aligned}$$

Then

$$\liminf_{t\to\infty} y(t) \geq \frac{\delta m}{\beta} \equiv \bar{l}$$

So, for large t, $y(t) > \frac{\bar{l}}{2} \equiv l > 0$. Let

$$D = \{(x,y) | m \leq x \leq M,\ l \leq y \leq L\}$$

Then D is bounded compact region in R_+^2 that has positive distance from coordinate hyperplanes. Hence we obtain that there exists a $T^* > 0$ such that if $t \geq T^*$, then every positive solution of system (2.1) with the initial conditions (2.2) eventually enters and remains in the region D, that is, system (2.1) is uniformly persistent.

Theorem 2.1 *If $p(x) = cx$ in the Holling-type I model, and the delay τ satisfy*

$$r - cL > 0 \tag{2.13}$$

$$\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mr^2x^*\tau}{K^2} - \frac{Mrcy^*\tau}{2K} > 0 \tag{2.14}$$

$$\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mrcy^*\tau}{2K} > 0 \tag{2.15}$$

where m, M, and L defined in Lemmas 2.2 and 2.3, then the unique positive equilibrium E^* of the system (2.1) is globally asymptotically stable.

Proof: Define $z(t) = (z_1(t), z_2(t))$ by

$$z_1(t) = \frac{x(t) - x^*}{x^*}, \quad z_2(t) = \frac{y(t) - y^*}{y^*}$$

From (2.1),

$$\dot{z}_1(t) = [1 + z_1(t)]\left[-\frac{rx^*}{K}z_1(t-\tau) - cy^*z_2(t)\right] \tag{2.16}$$

$$\dot{z}_2(t) = [1 + z_2(t)]\left\{\frac{\delta x^* z_1(t) - \beta y^* z_2(t)}{x^*[1 + z_1(t)]}\right\} \tag{2.17}$$

Let

$$V_1(z(t)) = \{z_1(t) - \ln[1+z_1(t)]\} + \{z_2(t) - \ln[1+z_2(t)]\} \tag{2.18}$$

then we have from (2.16) and (2.17) that

$$\begin{aligned}
\dot{V}_1(z(t)) &= \frac{z_1(t)\dot{z}_1(t)}{1+z_1(t)} + \frac{z_2(t)\dot{z}_2(t)}{1+z_2(t)} \\
&= -\frac{rx^*}{K} z_1(t) z_1(t-\tau) - cy^* z_1(t) z_2(t) + \frac{\delta}{1+z_1(t)} z_1(t) z_2(t) \\
&\quad - \frac{\beta y^* z_2^2(t)}{x^*[1+z_1(t)]} \\
&\leq -\frac{rx^*}{K} z_1(t) z_1(t-\tau) + \frac{\delta - cy^*[1+z_1(t)]}{1+z_1(t)} z_1(t) z_2(t) \\
&\quad - \frac{\beta y^* z_2^2(t)}{x^*[1+z_1(t)]}
\end{aligned} \tag{2.19}$$

If $\delta x^* - cy^* M > 0$, and by Lemma 2.3, there exists a $T^* > 0$ such that $m \leq x^*[1+z_1(t)] \leq M$ and $l \leq y^*[1+z_2(t)] \leq L$ for $t > T^*$. Then (2.19) implies that

$$\begin{aligned}
\dot{V}_1(z(t)) &\leq -\frac{rx^*}{K} z_1(t) z_1(t-\tau) \\
&\quad + \frac{\delta x^* - cy^* m}{2m}\left[z_1^2(t) + z_2^2(t)\right] - \frac{\beta y^* z_2^2(t)}{M} \\
&= -\frac{rx^*}{K} z_1(t)\left[z_1(t) - \int_{t-\tau}^{t} \dot{z}_1(s)\,ds\right] + \left(\frac{\delta x^*}{2m} - \frac{cy^*}{2}\right) z_1^2(t) \\
&\quad + \left(\frac{\delta x^*}{2m} - \frac{cy^*}{2} - \frac{\beta y^*}{M}\right) z_2^2(t) \\
&= -\left(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_1^2(t) - \left(\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_2^2(t) \\
&\quad + \frac{rx^*}{K} z_1(t) \int_{t-\tau}^{t} [1+z_1(s)]\left[-\frac{rx^*}{K} z_1(s-\tau) - cy^* z_2(s)\right] ds \\
&= -\left(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_1^2(t) - \left(\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_2^2(t) \\
&\quad + \frac{rx^*}{K} \int_{t-\tau}^{t} [1+z_1(s)]\left[-\frac{rx^*}{K} z_1(t) z_1(s-\tau) - cy^* z_1(t) z_2(s)\right] ds \\
&\leq -\left(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_1^2(t) - \left(\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_2^2(t) \\
&\quad + \frac{rx^*}{K} \int_{t-\tau}^{t} [1+z_1(s)]\left[\frac{rx^*}{K} |z_1(t)|\,|z_1(s-\tau)| + cy^* |z_1(t)|\,|z_2(s)|\right] ds
\end{aligned} \tag{2.20}$$

Then for $t \geq T^* + \tau \equiv \widehat{T}$, we have from (2.20) that

$$\begin{aligned}
\dot{V}_1(z(t)) \leq & -\left(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_1^2(t) - \left(\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_2^2(t) \\
& + \frac{Mr}{K}\int_{t-\tau}^{t}\left[\frac{rx^*}{K}|z_1(t)||z_1(s-\tau)| + cy^*|z_1(t)||z_2(s)|\right] ds \\
\leq & -\left(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_1^2(t) - \left(\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_2^2(t) \\
& + \frac{Mr}{K}\left[\frac{rx^*\tau}{2K} z_1^2(t) + \frac{rx^*}{2K}\int_{t-\tau}^{t} z_1^2(s-\tau)ds + \frac{cy^*\tau}{2} z_1^2(t) \right. \\
& \left. + \frac{cy^*}{2}\int_{t-\tau}^{t} z_2^2(s)ds\right] \\
= & -\left(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mr^2x^*\tau}{2K^2} - \frac{Mrcy^*\tau}{2K}\right) z_1^2(t) - \left(\frac{\beta y^*}{M}\right. \\
& \left. + \frac{cy^*}{2} - \frac{\delta x^*}{2m}\right) z_2^2(t) + \frac{Mr^2x^*}{2K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds + \frac{Mrcy^*}{2K}\int_{t-\tau}^{t} z_2^2(s)ds
\end{aligned} \tag{2.21}$$

Let

$$\begin{aligned}
V_2(z(t)) = & \frac{Mr^2x^*}{2K^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds \\
& + \frac{Mrcy^*}{2K}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds
\end{aligned} \tag{2.22}$$

then

$$\begin{aligned}
\dot{V}_2(z(t)) = & \frac{Mr^2x^*\tau}{2K^2} z_1^2(t-\tau) - \frac{Mr^2x^*}{2K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds \\
& + \frac{Mrcy^*\tau}{2K} z_2^2(t) - \frac{Mrcy^*}{2K}\int_{t-\tau}^{t} z_2^2(s)ds
\end{aligned} \tag{2.23}$$

and then we have from (2.21) and (2.23) that for $t \geq \widehat{T}$

$$\begin{aligned}
\dot{V}_1(z(t)) + \dot{V}_2(z(t)) \leq & -\left(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mr^2x^*\tau}{2K^2} - \frac{Mrcy^*\tau}{2K}\right) z_1^2(t) \\
& - \left(\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mrcy^*\tau}{2K}\right) z_2^2(t) \\
& + \frac{Mr^2x^*\tau}{2K^2} z_1^2(t-\tau)
\end{aligned} \tag{2.24}$$

Let

$$V_3(z(t)) = \frac{Mr^2x^*\tau}{2K^2}\int_{t-\tau}^{t} z_1^2(s)ds \tag{2.25}$$

then

$$\dot{V}_3(z(t)) = \frac{Mr^2x^*\tau}{2K^2}z_1^2(t) - \frac{Mr^2x^*\tau}{2K^2}z_1^2(t-\tau) \tag{2.26}$$

Now define a Lyapunov functional $V(z(t))$ as

$$V(z(t)) = V_1(z(t)) + V_2(z(t)) + V_3(z(t)) \tag{2.27}$$

then we have from (2.24) and (2.26) that for $t \geq \widehat{T}$

$$\begin{aligned} \dot{V}(z(t)) &\leq -\left(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mr^2x^*\tau}{K^2} - \frac{Mrcy^*\tau}{2K}\right)z_1^2(t) \\ &\quad -\left(\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mrcy^*\tau}{2K}\right)z_2^2(t) \\ &\equiv -\zeta z_1^2(t) - \eta z_2^2(t) \end{aligned} \tag{2.28}$$

Then it follows from (2.14) and (2.15) that $\zeta > 0$ and $\eta > 0$. Let $w(s) = \widehat{N}s^2$ where $\widehat{N} = \min\{\zeta, \eta\}$, then w is nonnegative continuous on $[0,\infty)$, $w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (2.28) that for $t \geq \widehat{T}$

$$\dot{V}(z(t)) \leq -\widehat{N}\left[z_1^2(t) + z_2^2(t)\right] = -\widehat{N}\|z(t)\|^2 = -w(\|z(t)\|) \tag{2.29}$$

Now, we want to find a function u such that $V(z(t)) \geq u(\|z(t)\|)$. It follows from (2.18), (2.22), and (2.25) that

$$V(z(t)) \geq \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \tag{2.30}$$

By the Taylor Theorem, we have that

$$z_i(t) - \ln[1 + z_i(t)] = \frac{z_i^2(t)}{2[1 + \theta_i(t)]^2} \tag{2.31}$$

where $\theta_i(t) \in (0, z_i(t))$ or $(z_i(t), 0)$ for $i = 1, 2$.

Case1 : If $0 < \theta_i(t) < z_i(t)$ for $i = 1, 2$, then

$$\frac{z_i^2(t)}{[1 + z_i(t)]^2} < \frac{z_i^2(t)}{[1 + \theta_i(t)]^2} < z_i^2(t) \tag{2.32}$$

By Lemma 2.3, it follows that for $t \geq T^*$

$$\begin{aligned} m \leq x^*[1 + z_1(t)] &= x(t) \leq M \\ l \leq y^*[1 + z_2(t)] &= y(t) \leq L \end{aligned} \tag{2.33}$$

Then (2.32) implies that

$$
\begin{aligned}
\left(\frac{x^*}{M}\right)^2 z_1^2(t) &\le \frac{z_1^2(t)}{[1+\theta_1(t)]^2} < z_1^2(t) \\
\left(\frac{y^*}{L}\right)^2 z_2^2(t) &\le \frac{z_2^2(t)}{[1+\theta_2(t)]^2} < z_2^2(t)
\end{aligned}
\tag{2.34}
$$

It follows that (2.30), (2.31), and (2.34) that for $t \ge T^*$

$$
\begin{aligned}
V(z(t)) &\ge \frac{1}{2}\frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2}\frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\
&\ge \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t) \\
&\ge \min\left\{\frac{1}{2}\left(\frac{x^*}{M}\right)^2, \frac{1}{2}\left(\frac{y^*}{L}\right)^2\right\} \cdot [z_1^2(t) + z_2^2(t)] \\
&\equiv \widetilde{N}\, \|z(t)\|^2
\end{aligned}
$$

Case2 : If $-1 < z_i(t) < \theta_i(t) < 0$ for $i = 1,2$, then

$$
z_i^2(t) < \frac{z_i^2(t)}{[1+\theta_i(t)]^2} < \frac{z_i^2(t)}{[1+z_i(t)]^2}
\tag{2.35}
$$

By (2.33), (2.35) implies that

$$
\begin{aligned}
z_1^2(t) &< \frac{z_1^2(t)}{[1+\theta_1(t)]^2} \le \left(\frac{x^*}{m}\right)^2 z_1^2(t) \\
z_2^2(t) &< \frac{z_2^2(t)}{[1+\theta_2(t)]^2} \le \left(\frac{y^*}{l}\right)^2 z_2^2(t)
\end{aligned}
\tag{2.36}
$$

It follows that (2.30), (2.31), and (2.36) that for $t \ge T^*$

$$
\begin{aligned}
V(z(t)) &\ge \frac{1}{2}\frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2}\frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\
&> \frac{1}{2}z_1^2(t) + \frac{1}{2}z_2^2(t) \\
&\ge \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t) \\
&\ge \widetilde{N}\,[z_1^2(t) + z_2^2(t)] \\
&= \widetilde{N}\|z(t)\|^2
\end{aligned}
$$

Case3 : If $0 < \theta_1(t) < z_1(t)$ and $-1 < z_2(t) < \theta_2(t) < 0$, then it follows that (2.30), (2.31), (2.34), and (2.36) that for $t \geq T^*$

$$\begin{aligned} V(z(t)) &\geq \frac{1}{2}\frac{z_1^2(t)}{[1+\theta_1(t)]^2}+\frac{1}{2}\frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\ &> \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t)+\frac{1}{2}z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t)+\frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t) \\ &\geq \tilde{N}\left[z_1^2(t)+z_2^2(t)\right] \\ &= \tilde{N}\|z(t)\|^2 \end{aligned}$$

Case4 : If $-1 < z_1(t) < \theta_1(t) < 0$ and $0 < \theta_2(t) < z_2(t)$, then it follows that (2.30), (2.31), (2.34), and (2.36) that for $t \geq T^*$

$$\begin{aligned} V(y(t)) &\geq \frac{1}{2}\frac{z_1^2(t)}{[1+\theta_1(t)]^2}+\frac{1}{2}\frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\ &> \frac{1}{2}z_1^2(t)+\frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t)+\frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t) \\ &\geq \tilde{N}\left[z_1^2(t)+z_2^2(t)\right] \\ &= \tilde{N}\|z(t)\|^2 \end{aligned}$$

Let $u(s) = \tilde{N}s^2$, then u is nonnegative continuous on $[0,\infty)$, $u(0) = 0$, $u(s) > 0$ for $s > 0$, and $\lim_{s\to+\infty} u(s) = +\infty$. So, by case1 $\sim$ case4, we have

$$V(z(t)) \geq u(\|z(t)\|) \qquad \text{for } t \geq T^* \tag{2.37}$$

So the equilibrium point E^* of the system (2.1) is globally asymptotically stable with $p(x) = cx$.

Theorem 2.2 *If $p(x) = \frac{cx}{1+x}$ in the Holling-type II model, and the delay τ satisfy*

$$r - cL > 0 \tag{2.38}$$

$$\begin{aligned} &\frac{rx^*(1+2x^*)}{K(1+M)}+\frac{cy^*}{2(1+m)}-\frac{rx^*}{1+m}-\frac{\delta x^*(1+M)}{2m(1+m)}-\frac{rx^*(x^*+M)}{K(1+m)} \\ &-\frac{Mr^2x^*\tau(1+x^*)(K+3x^*+1+M)}{K^2(1+m)^2}-\frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2} > 0 \end{aligned} \tag{2.39}$$

$$\frac{\beta y^*}{M}+\frac{cy^*}{2(1+m)}-\frac{\delta x^*(1+M)}{2m(1+m)}-\frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}>0 \tag{2.40}$$

where m, M, and L defined in Lemmas 2.2 and 2.3, then the unique positive equilibrium E^* of the system (2.1) is globally asymptotically stable.

Proof: Define $z(t)=(z_1(t),z_2(t))$ by

$$z_1(t)=\frac{x(t)-x^*}{x^*}\quad,\quad z_2(t)=\frac{y(t)-y^*}{y^*}$$

From (2.1),

$$\begin{aligned}\dot{z}_1(t) &= [1+z_1(t)]\left\{\frac{rx^*z_1(t)}{1+x^*[1+z_1(t)]}-\frac{rx^{*2}z_1(t)}{K\{1+x^*[1+z_1(t)]\}}\right.\\ &\quad -\frac{rx^*z_1(t-\tau)}{K\{1+x^*[1+z_1(t)]\}}-\frac{rx^{*2}z_1(t-\tau)}{K\{1+x^*[1+z_1(t)]\}}\\ &\quad \left.-\frac{rx^{*2}z_1(t-\tau)z_1(t)}{K\{1+x^*[1+z_1(t)]\}}-\frac{cy^*z_2(t)}{1+x^*[1+z_1(t)]}\right\}\end{aligned} \tag{2.41}$$

$$\dot{z}_2(t) = [1+z_2(t)]\left\{\frac{\delta x^*z_1(t)-\beta y^*z_2(t)}{x^*[1+z_1(t)]}\right\} \tag{2.42}$$

Let

$$V_1(z(t))=\{z_1(t)-\ln[1+z_1(t)]\}+\{z_2(t)-\ln[1+z_2(t)]\} \tag{2.43}$$

then we have from (2.41) and (2.42) that

$$\begin{aligned}\dot{V}_1(z(t)) &= \frac{\dot{z}_1(t)z_1(t)}{1+z_1(t)}+\frac{\dot{z}_2(t)z_2(t)}{1+z_2(t)}\\ &= \frac{rx^*z_1^2(t)}{1+x^*[1+z_1(t)]}-\frac{rx^{*2}z_1^2(t)}{K\{1+x^*[1+z_1(t)]\}}-\frac{rx^*z_1(t)z_1(t-\tau)}{K\{1+x^*[1+z_1(t)]\}}\\ &\quad -\frac{rx^{*2}z_1(t)z_1(t-\tau)}{K\{1+x^*[1+z_1(t)]\}}-\frac{rx^{*2}z_1^2(t)z_1(t-\tau)}{K\{1+x^*[1+z_1(t)]\}}-\frac{\beta y^*z_2^2(t)}{x^*[1+z_1(t)]}\\ &\quad +\frac{\delta\{1+x^*[1+z_1(t)]\}-cy^*[1+z_1(t)]}{\{1+x^*[1+z_1(t)]\}[1+z_1(t)]}z_1(t)z_2(t)\\ &\le \frac{rx^*z_1^2(t)}{1+x^*[1+z_1(t)]}-\frac{rx^{*2}z_1^2(t)}{K\{1+x^*[1+z_1(t)]\}}+\frac{rx^{*2}z_1^2(t)|z_1(t-\tau)|}{K\{1+x^*[1+z_1(t)]\}}\\ &\quad -\frac{\beta y^*z_2^2(t)}{x^*[1+z_1(t)]}+\frac{\delta\{1+x^*[1+z_1(t)]\}-cy^*[1+z_1(t)]}{\{1+x^*[1+z_1(t)]\}[1+z_1(t)]}z_1(t)z_2(t)\\ &\quad -\left\{\frac{rx^*}{K\{1+x^*[1+z_1(t)]\}}+\frac{rx^{*2}}{K\{1+x^*[1+z_1(t)]\}}\right\}z_1(t)z_1(t-\tau)\end{aligned} \tag{2.44}$$

If $\delta x^*(1+m) - cy^*M > 0$, and by Lemma 2.3, there exists a $T^* > 0$ such that $m \le x^*[1+z_1(t)] \le M$ and $l \le y^*[1+z_2(t)] \le L$ for $t > T^*$. Then (2.44) implies that

$$\begin{aligned}
\dot{V}_1(z(t)) \le\; & \frac{rx^* z_1^2(t)}{1+m} - \frac{rx^{*2} z_1^2(t)}{K(1+M)} + \frac{rx^{*2} z_1^2(t)\,|z_1(t-\tau)|}{K(1+m)} \\
& - \frac{\beta y^* z_2^2(t)}{M} - \frac{cy^* z_1^2(t)}{2(1+m)} - \frac{cy^* z_2^2(t)}{2(1+m)} + \frac{\delta x^*(1+M) z_1^2(t)}{2m(1+m)} \\
& + \frac{\delta x^*(1+M) z_2^2(t)}{2m(1+m)} - \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} z_1(t)\left[z_1(t) - \int_{t-\tau}^{t} \dot{z}_1(s)ds\right] \\
=\; & -\left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)}\right] z_1^2(t) \\
& - \left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)}\right] z_2^2(t) + \frac{rx^{*2}}{K(1+m)} z_1^2(t)\,|z_1(t-\tau)| \\
& + \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} \int_{t-\tau}^{t} [1+z_1(s)] \left\{ \frac{rx^* z_1(t) z_1(s)}{1+x^*[1+z_1(s)]} \right. \\
& - \frac{rx^{*2} z_1(t) z_1(s)}{K\{1+x^*[1+z_1(s)]\}} - \frac{rx^* z_1(t) z_1(s-\tau)}{K\{1+x^*[1+z_1(s)]\}} - \frac{rx^{*2} z_1(t) z_1(s-\tau)}{K\{1+x^*[1+z_1(s)]\}} \\
& \left. - \frac{rx^{*2} z_1(t) z_1(s) z_1(s-\tau)}{K\{1+x^*[1+z_1(s)]\}} - \frac{cy^* z_1(t) z_2(s)}{1+x^*[1+z_1(s)]} \right\} ds \\
\le\; & -\left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)}\right] z_1^2(t) \\
& - \left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)}\right] z_2^2(t) + \frac{rx^{*2}}{K(1+m)} z_1^2(t)\,|z_1(t-\tau)| \\
& + \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} \int_{t-\tau}^{t} [1+z_1(s)] \left\{ \frac{rx^* |z_1(t)|\,|z_1(s)|}{1+x^*[1+z_1(s)]} \right. \\
& + \frac{rx^{*2} |z_1(t)|\,|z_1(s)|}{K\{1+x^*[1+z_1(s)]\}} + \frac{rx^* |z_1(t)|\,|z_1(s-\tau)|}{K\{1+x^*[1+z_1(s)]\}} \\
& \left. + \frac{rx^{*2} |z_1(t)|\,|z_1(s-\tau)|}{K\{1+x^*[1+z_1(s)]\}} + \frac{cy^* |z_1(t)|\,|z_2(s)|}{1+x^*[1+z_1(s)]} \right\} ds \\
& - \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} \int_{t-\tau}^{t} \frac{rx^{*2} z_1(t) z_1(s-\tau)}{K\{1+x^*[1+z_1(s)]\}} [1+z_1(s)]^2 ds \\
& + \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} \int_{t-\tau}^{t} \frac{rx^{*2} |z_1(t)|\,|z_1(s-\tau)|}{K\{1+x^*[1+z_1(s)]\}} [1+z_1(s)] ds \qquad (2.45)
\end{aligned}$$

Then for $t \geq T^* + \tau \equiv \widehat{T}$, we have from (2.45) that

$$
\begin{aligned}
\dot{V}_1(z(t)) \leq & -\left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)}\right] z_1^2(t) \\
& -\left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)}\right] z_2^2(t) + \frac{rx^{*2}}{K(1+m)}\left(1+\frac{M}{x^*}\right) z_1^2(t) \\
& + \frac{Mr(1+x^*)}{K(1+m)} \int_{t-\tau}^{t} \left[\frac{rx^*|z_1(t)||z_1(s)|}{1+m} + \frac{rx^{*2}|z_1(t)||z_1(s)|}{K(1+m)}\right. \\
& \left. + \frac{rx^*|z_1(t)||z_1(s-\tau)|}{K(1+m)} + \frac{rx^{*2}|z_1(t)||z_1(s-\tau)|}{K(1+m)} + \frac{cy^*|z_1(t)||z_2(s)|}{1+m}\right] ds \\
& + \frac{M^2 r(1+x^*)}{Kx^*(1+m)} \int_{t-\tau}^{t} \frac{rx^{*2}|z_1(t)||z_1(s-\tau)|}{K(1+m)} ds \\
& + \frac{Mr(1+x^*)}{K(1+m)} \int_{t-\tau}^{t} \frac{rx^{*2}|z_1(t)||z_1(s-\tau)|}{K(1+m)} ds \\
\leq & -\left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)}\right. \\
& - \frac{rx^*(x^*+M)}{K(1+m)} - \frac{Mr(1+x^*)\tau}{K(1+m)^2}\left(\frac{rx^*}{2} + \frac{3rx^{*2}}{2K} + \frac{rx^*}{2K} + \frac{cy^*}{2}\right) \\
& \left. - \frac{M^2 r^2 x^*(1+x^*)\tau}{2K^2(1+m)^2}\right] z_1^2(t) - \left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*}{2m}\right] z_2^2(t) \\
& + \frac{Mr^2 x^*(1+x^*)(K+x^*)}{2K^2(1+m)^2} \int_{t-\tau}^{t} z_1^2(s) ds \\
& + \frac{Mr^2 x^*(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2} \int_{t-\tau}^{t} z_1^2(s-\tau) ds \\
& + \frac{Mrcy^*(1+x^*)}{2K(1+m)^2} \int_{t-\tau}^{t} z_2^2(s) ds \qquad (2.46)
\end{aligned}
$$

Let

$$
\begin{aligned}
V_2(z(t)) = & \frac{Mr^2 x^*(1+x^*)(K+x^*)}{2K^2(1+m)^2} \int_{t-\tau}^{t} \int_{s}^{t} z_1^2(\gamma) d\gamma ds \\
& + \frac{Mr^2 x^*(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2} \int_{t-\tau}^{t} \int_{s}^{t} z_1^2(\gamma-\tau) d\gamma ds \\
& + \frac{Mrcy^*(1+x^*)}{2K(1+m)^2} \int_{t-\tau}^{t} \int_{s}^{t} z_2^2(\gamma) d\gamma ds \qquad (2.47)
\end{aligned}
$$

then

$$\begin{aligned}\dot{V}_2(z(t)) &= \frac{Mr^2x^*\tau(1+x^*)(K+x^*)}{2K^2(1+m)^2}z_1^2(t)\\ &-\frac{Mr^2x^*(1+x^*)(K+x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t}z_1^2(s)ds\\ &+\frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}z_1^2(t-\tau)\\ &-\frac{Mr^2x^*(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t}z_1^2(s-\tau)ds\\ &+\frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}z_2^2(t)-\frac{Mrcy^*(1+x^*)}{2K(1+m)^2}\int_{t-\tau}^{t}z_2^2(s)ds\end{aligned} \tag{2.48}$$

and then we have from (2.46) and (2.48) that for $t \geq \widehat{T}$

$$\begin{aligned}\dot{V}_1(z(t))+\dot{V}_2(z(t)) \leq &-\left[\frac{rx^*(1+2x^*)}{K(1+M)}+\frac{cy^*}{2(1+m)}-\frac{rx^*}{1+m}-\frac{\delta x^*(1+M)}{2m(1+m)}\right.\\ &-\frac{rx^*(x^*+M)}{K(1+m)}-\frac{Mr(1+x^*)\tau}{K(1+m)^2}\left(\frac{rx^*}{2}+\frac{3rx^{*2}}{2K}+\frac{rx^*}{2K}+\frac{cy^*}{2}\right)\\ &\left.-\frac{M^2r^2x^*\tau(1+x^*)}{2K^2(1+m)^2}-\frac{Mr^2x^*\tau(1+x^*)(K+x^*)}{2K^2(1+m)^2}\right]z_1^2(t)\\ &-\left[\frac{\beta y^*}{M}+\frac{cy^*}{2(1+m)}-\frac{\delta x^*(1+M)}{2m(1+m)}-\frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}\right]z_2^2(t)\\ &+\frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}z_1^2(t-\tau)\end{aligned} \tag{2.49}$$

Let

$$V_3(z(t)) = \frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t}z_1^2(s)ds \tag{2.50}$$

then

$$\begin{aligned}\dot{V}_3(z(t)) &= \frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}z_1^2(t)\\ &-\frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}z_1^2(t-\tau)\end{aligned} \tag{2.51}$$

Now define a Lyapunov functional $V(z(t))$ as

$$V(z(t)) = V_1(z(t)) + V_2(z(t)) + V_3(z(t)) \tag{2.52}$$

then we have from (2.49) and (2.51) that for $t \geq \widehat{T}$

$$\begin{aligned}
\dot{V}(z(t)) \leq & -\left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)}\right. \\
& -\frac{rx^*(x^*+M)}{K(1+m)} - \frac{Mr^2x^*\tau(1+x^*)(K+3x^*+1+M)}{K^2(1+m)^2} \\
& \left.-\frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}\right] z_1^2(t) \\
& -\left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}\right] z_2^2(t) \\
\equiv & -\zeta z_1^2(t) - \eta z_2^2(t)
\end{aligned} \tag{2.53}$$

Then it follows from (2.39) and (2.40) that $\zeta > 0$ and $\eta > 0$. Let $w(s) = \widehat{N}s^2$ where $\widehat{N} = \min\{\zeta, \eta\}$, then w is nonnegative continuous on $[0, \infty)$, $w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (2.53) that for $t \geq \widehat{T}$

$$\dot{V}(z(t)) \leq -\widehat{N}\left[z_1^2(t) + z_2^2(t)\right] = -\widehat{N}\|z(t)\|^2 = -w(\|z(t)\|) \tag{2.54}$$

Now, we want to find a function u such that $V(z(t)) \geq u(\|z(t)\|)$. It follows from (2.43), (2.47), and (2.50) that

$$V(z(t)) \geq \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \tag{2.55}$$

then by Theorem 2.1, we have that $u(s) = \widetilde{N}s^2$ and $V(z(t)) \geq u(\|z(t)\|)$. So the equilibrium point E^* of the system (2.1) is globally asymptotically stable with $p(x) = \frac{cx}{1+x}$.

Theorem 2.3 *If $p(x) = \frac{cx^2}{1+x^2}$ in the Holling-type III model, and the delay τ satisfy*

$$r - cL > 0 \tag{2.56}$$

$$\begin{aligned}
&\frac{rx^*(1+3x^{*2})}{K(1+M^2)}+\frac{cx^*y^*}{2(1+m^2)}+\frac{cx^*y^*}{1+M^2}-\frac{rx^*(M+3x^*)}{1+m^2}-\frac{3rx^{*2}(M+x^*)}{K(1+m^2)}\\
&-\frac{rx^*(M+x^*)^2}{K(1+m^2)}-\frac{\delta x^*(1+M^2)}{2m(1+m^2)}-\frac{cx^*(L+y^*)}{1+m^2}\\
&-\frac{Mr^2x^*\tau(1+x^{*2})(3Kx^*+9x^{*2}+MK+5Mx^*+1+M^2)}{K^2(1+m^2)^2}\\
&-\frac{Mrcy^*\tau(1+x^{*2})(4x^*+M)}{2K(1+m^2)^2}>0
\end{aligned}\tag{2.57}$$

$$\frac{\beta y^*}{M}+\frac{cx^*y^*}{2(1+m^2)}-\frac{\delta x^*(1+M^2)}{2m(1+m^2)}-\frac{Mrcy^*\tau(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2}>0\tag{2.58}$$

where m, M, and L defined in Lemmas 2.2 and 2.3, then the unique positive equilibrium E^* of the system (2.1) is globally asymptotically stable.

Proof: Define $z(t)=(z_1(t),z_2(t))$ by

$$z_1(t)=\frac{x(t)-x^*}{x^*}\ ,\ z_2(t)=\frac{y(t)-y^*}{y^*}$$

From (2.1),

$$\begin{aligned}
\dot z_1(t) \;=\; &[1+z_1(t)]\left\{\frac{2rx^{*2}z_1(t)}{1+x^{*2}[1+z_1(t)]^2}+\frac{rx^{*2}z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2}\right.\\
&-\frac{2rx^{*3}z_1(t)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}-\frac{rx^{*3}z_1^2(t)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}\\
&-\frac{rx^*z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}-\frac{rx^{*3}z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}\\
&-\frac{2rx^{*3}z_1(t)z_1(t-\tau)}{K\left\{1+x_1^{*2}[1+z_1(t)]^2\right\}}-\frac{rx^{*3}z_1^2(t)z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}\\
&\left.-\frac{cx^*y^*z_1(t)}{1+x^{*2}[1+z_1(t)]^2}-\frac{cx^*y^*z_2(t)}{1+x^{*2}[1+z_1(t)]^2}-\frac{cx^*y^*z_1(t)z_2(t)}{1+x^{*2}[1+z_1(t)]^2}\right\}
\end{aligned}\tag{2.59}$$

$$\dot z_2(t) \;=\; [1+z_2(t)]\left\{\frac{\delta x^*z_1(t)-\beta y^*z_2(t)}{x^*[1+z_1(t)]}\right\}\tag{2.60}$$

Let

$$V_1(z(t)) = \{z_1(t) - \ln[1+z_1(t)]\} + \{z_2(t) - \ln[1+z_2(t)]\} \tag{2.61}$$

then we have from (2.59) and (2.60) that

$$\begin{aligned}
\dot{V}_1(z(t)) &= \frac{\dot{z}_1(t)z_1(t)}{1+z_1(t)} + \frac{\dot{z}_2(t)z_2(t)}{1+z_2(t)} \\
&= \frac{2rx^{*2}z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2} + \frac{rx^{*2}z_1^3(t)}{1+x^{*2}[1+z_1(t)]^2} \\
&\quad - \frac{2rx^{*3}z_1^2(t)}{K\left\{1+x_1^{*2}[1+z_1(t)]^2\right\}} - \frac{rx^{*3}z_1^3(t)}{K\left\{1+x_1^{*2}[1+z_1(t)]^2\right\}} \\
&\quad - \frac{rx^{*}z_1(t)z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}} - \frac{rx^{*3}z_1(t)z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}} \\
&\quad - \frac{2rx^{*3}z_1^2(t)z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}} - \frac{rx^{*3}z_1^3(t)z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}} \\
&\quad - \frac{cx^{*}y^{*}z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2} - \frac{\beta y^{*}z_2^2(t)}{x^{*}[1+z_1(t)]} \\
&\quad - \frac{cx^{*}y^{*}z_1^2(t)z_2(t)}{1+x^{*2}[1+z_1(t)]^2} \\
&\quad + \frac{\delta\left\{1+x^{*2}[1+z_1(t)]^2\right\} - cx^{*}y^{*}[1+z_1(t)]}{\left\{1+x^{*2}[1+z_1(t)]^2\right\}[1+z_1(t)]} z_1(t)z_2(t)
\end{aligned} \tag{2.62}$$

If $\delta(1+m^2) - cy^{*}M > 0$, and by Lemma 2.3, there exists a $T^{*} > 0$ such that $m \le x^{*}[1+z_1(t)] \le M$ and $l \le y^{*}[1+z_2(t)] \le L$ for $t > T^{*}$. Then (2.62) implies that

$$
\begin{aligned}
\dot V_1(z(t)) \le\ & \frac{2rx^{*2}z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2}-\frac{2rx^{*3}z_1^2(t)}{K\left\{1+x_1^{*2}[1+z_1(t)]^2\right\}}-\frac{cx^*y^*z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2}\\
&+\frac{rx^{*2}z_1^2(t)\,|z_1(t)|}{1+x^{*2}[1+z_1(t)]^2}+\frac{rx^{*3}z_1^2(t)\,|z_1(t)|}{K\left\{1+x_1^{*2}[1+z_1(t)]^2\right\}}+\frac{2rx^{*3}z_1^2(t)\,|z_1(t-\tau)|}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}\\
&+\frac{cx^*y^*z_1^2(t)\,|z_2(t)|}{1+x^{*2}[1+z_1(t)]^2}+\frac{rx^{*3}z_1^2(t)\,|z_1(t)|\,|z_1(t-\tau)|}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}-\frac{\beta y^*z_2^2(t)}{x^*[1+z_1(t)]}\\
&+\frac{\delta\left\{1+x^{*2}[1+z_1(t)]^2\right\}-cx^*y^*[1+z_1(t)]}{\left\{1+x^{*2}[1+z_1(t)]^2\right\}[1+z_1(t)]}|z_1(t)|\,|z_2(t)|\\
&-\frac{rx^*z_1(t)z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}-\frac{rx^{*3}z_1(t)z_1(t-\tau)}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}\\
\le\ & \frac{2rx^{*2}}{1+m^2}z_1^2(t)-\frac{2rx^{*3}}{K(1+M^2)}z_1^2(t)-\frac{cx^*y^*}{1+M^2}z_1^2(t)+\frac{rx^{*2}}{1+m^2}z_1^2(t)\,|z_1(t)|\\
&+\frac{rx^{*3}}{K(1+m^2)}z_1^2(t)\,|z_1(t)|+\frac{2rx^{*3}}{K(1+m^2)}z_1^2(t)\,|z_1(t-\tau)|\\
&+\frac{cx^*y^*}{1+m^2}z_1^2(t)\,|z_2(t)|+\frac{rx^{*3}}{K(1+m^2)}z_1(t)^2\,|z_1(t)|\,|z_1(t-\tau)|-\frac{\beta y^*}{M}z_2^2(t)\\
&+\frac{\delta x^*(1+M^2)}{2m(1+m^2)}\left[z_1^2(t)+z_2^2(t)\right]-\frac{cx^*y^*}{2(1+m^2)}\left[z_1^2(t)+z_2^2(t)\right]\\
&-\frac{rx^*(1+x^{*2})}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}z_1(t)\left[z_1(t)-\int_{t-\tau}^{t}\dot z_1(s)ds\right]\\
\le\ & \frac{2rx^{*2}}{1+m^2}z_1^2(t)-\frac{2rx^{*3}}{K(1+M^2)}z_1^2(t)-\frac{cx^*y^*}{1+M^2}z_1^2(t)+\frac{rx^*(M+x^*)}{1+m^2}z_1^2(t)\\
&+\frac{rx^{*2}(M+x^*)}{K(1+m^2)}z_1^2(t)+\frac{cx^*(L+y^*)}{1+m^2}z_1^2(t)-\frac{\beta y^*}{M}z_2^2(t)+\frac{\delta x^*(1+M^2)}{2m(1+m^2)}z_1^2(t)\\
&-\frac{cx^*y^*}{2(1+m^2)}z_1^2(t)+\frac{\delta x^*(1+M^2)}{2m(1+m^2)}z_2^2(t)-\frac{cx^*y^*}{2(1+m^2)}z_2^2(t)\\
&-\frac{rx^*(1+x^{*2})}{K(1+M^2)}z_1^2(t)+\frac{2rx^{*3}}{K(1+m^2)}z_1^2(t)\,|z_1(t-\tau)|\\
&+\frac{rx^{*3}}{K(1+m^2)}z_1(t)^2\,|z_1(t)|\,|z_1(t-\tau)|\\
&+\frac{rx^*(1+x^{*2})}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}}\int_{t-\tau}^{t}[1+z_1(s)]\\
&\times\left\{\frac{2rx^{*2}z_1(t)z_1(s)}{1+x^{*2}[1+z_1(s)]^2}+\frac{rx^{*2}z_1(t)z_1^2(s)}{1+x^{*2}[1+z_1(s)]^2}-\frac{2rx^{*3}z_1(t)z_1(s)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}}\right.\\
&-\frac{rx^{*3}z_1(t)z_1^2(s)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}}-\frac{rx^*z_1(t)z_1(s-\tau)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}}\\
&-\frac{rx^{*3}z_1(t)z_1(s-\tau)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}}-\frac{2rx^{*3}z_1(t)z_1(s)z_1(s-\tau)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{rx^{*3} z_1(t) z_1^2(s) z_1(s-\tau)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}} - \frac{cx^* y^* z_1(t) z_1(s)}{1+x^{*2}[1+z_1(s)]^2} \\
&- \frac{cx^* y^* z_1(t) z_2(s)}{1+x^{*2}[1+z_1(s)]^2} - \frac{cx^* y^* z_1(t) z_1(s) z_2(s)}{1+x^{*2}[1+z_1(s)]^2} \Bigg\} ds
\end{aligned}
\tag{2.63}
$$

Then for $t \geq T^* + \tau \equiv \widehat{T}$, we have from (2.63) that

$$
\begin{aligned}
\dot{V}_1(z(t)) \leq\ & -\frac{cx^* y^*}{1+M^2} z_1^2(t) + \frac{rx^*(M+3x^*)}{1+m^2} z_1^2(t) + \frac{3rx^{*2}(M+x^*)}{K(1+m^2)} z_1^2(t) \\
& + \frac{cx^*(L+y^*)}{1+m^2} z_1^2(t) + \frac{rx^*(M+x^*)^2}{K(1+m^2)} z_1^2(t) - \frac{\beta y^*}{M} z_2^2(t) \\
& + \frac{\delta x^*(1+M^2)}{2m(1+m^2)} z_1^2(t) - \frac{cx^* y^*}{2(1+m^2)} z_1^2(t) - \frac{rx^*(1+3x^{*2})}{K(1+M^2)} z_1^2(t) \\
& + \frac{\delta x^*(1+M^2)}{2m(1+m^2)} z_2^2(t) - \frac{cx^* y^*}{2(1+m^2)} z_2^2(t) \\
& + \frac{Mr(1+x^{*2})}{K(1+m^2)^2} \int_{t-\tau}^{t} \Bigg(2rx^{*2} |z_1(t)||z_1(s)| + \frac{2rx^{*3}}{K} |z_1(t)||z_1(s)| \\
& + \frac{rx^*}{K} |z_1(t)||z_1(s-\tau)| + cx^* y^* |z_1(t)||z_2(s)| + cx^* y^* |z_1(t)||z_1(s)| \\
& \qquad\qquad + \frac{rx^{*3}}{K} |z_1(t)||z_1(s-\tau)| \Bigg) dsr \\
& + \frac{rx^*(1+x^{*2})}{K\left\{1+x^{*2}[1+z_1(t)]^2\right\}} \int_{t-\tau}^{t} [1+z_1(s)] \\
& \times \Bigg\{ \Bigg[\frac{rx^{*2} z_1(t) z_1(s)}{1+x^{*2}[1+z_1(s)]^2} - \frac{rx^{*3} z_1(t) z_1(s)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}} \\
& - \frac{2rx^{*3} z_1(t) z_1(s-\tau)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}} - \frac{cx^* y^* z_1(t) z_2(s)}{1+x^{*2}[1+z_1(s)]^2} \Bigg] [1+z_1(s)] \\
& - \Bigg[\frac{rx^{*2} z_1(t) z_1(s)}{1+x^{*2}[1+z_1(s)]^2} - \frac{rx^{*3} z_1(t) z_1(s)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}} \\
& \qquad - \frac{2rx^{*3} z_1(t) z_1(s-\tau)}{K\left\{1+x^{*2}[1+z_1(s)]^2\right\}} - \frac{cx^* y^* z_1(t) z_2(s)}{1+x^{*2}[1+z_1(s)]^2} \Bigg] \Bigg\} ds
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{rx^*\left(1+x^{*2}\right)}{K\left\{1+x^{*2}\left[1+z_1(t)\right]^2\right\}}\int_{t-\tau}^{t}\left[1+z_1(s)\right] \\
& \times\left\{\frac{rx^{*3}z_1(t)z_1(s-\tau)}{K\left\{1+x^{*2}\left[1+z_1(s)\right]^2\right\}}\left[1+z_1(s)\right]^2\right. \\
& \qquad\qquad \left.-\frac{rx^{*3}z_1(t)z_1(s-\tau)}{K\left\{1+x^{*2}\left[1+z_1(s)\right]^2\right\}}-\frac{2rx^{*3}z_1(t)z_1(s)z_1(s-\tau)}{K\left\{1+x^{*2}\left[1+z_1(s)\right]^2\right\}}\right\}ds \\
\leq\; & -\frac{cx^*y^*}{1+M^2}z_1^2(t)+\frac{rx^*\left(M+3x^*\right)}{1+m^2}z_1^2(t)+\frac{3rx^{*2}\left(M+x^*\right)}{K\left(1+m^2\right)}z_1^2(t) \\
& +\frac{cx^*\left(L+y^*\right)}{1+m^2}z_1^2(t)+\frac{rx^*\left(M+x^*\right)^2}{K\left(1+m^2\right)}z_1^2(t)-\frac{\beta y^*}{M}z_2^2(t) \\
& +\frac{\delta x^*\left(1+M^2\right)}{2m\left(1+m^2\right)}z_1^2(t)-\frac{cx^*y^*}{2\left(1+m^2\right)}z_1^2(t)-\frac{rx^*\left(1+3x^{*2}\right)}{K\left(1+M^2\right)}z_1^2(t) \\
& +\frac{\delta x^*\left(1+M^2\right)}{2m\left(1+m^2\right)}z_2^2(t)-\frac{cx^*y^*}{2\left(1+m^2\right)}z_2^2(t) \\
& +\frac{Mr\left(1+x^{*2}\right)}{K\left(1+m^2\right)^2}\left[rx^{*2}\tau z_1^2(t)+rx^{*2}\int_{t-\tau}^{t}z_1^2(s)ds+\frac{rx^{*3}\tau}{K}z_1^2(t)\right. \\
& +\frac{rx^{*3}}{K}\int_{t-\tau}^{t}z_1^2(s)ds+\frac{rx^*\tau}{2K}z_1^2(t)+\frac{rx^*}{2K}\int_{t-\tau}^{t}z_1^2(s-\tau)ds+\frac{rx^{*3}\tau}{2K}z_1^2(t) \\
& +\frac{rx^{*3}}{2K}\int_{t-\tau}^{t}z_1^2(s-\tau)ds+\frac{cx^*y^*\tau}{2}z_1^2(t)+\frac{cx^*y^*}{2}\int_{t-\tau}^{t}z_1^2(s)ds \\
& \left.+\frac{cx^*y^*\tau}{2}z_1^2(t)+\frac{cx^*y^*}{2}\int_{t-\tau}^{t}z_2^2(s)ds\right] \\
& +\frac{M^2r\left(1+x^{*2}\right)}{Kx^*\left(1+m^2\right)^2}\int_{t-\tau}^{t}\left[rx^{*2}\left|z_1(t)\right|\left|z_1(s)\right|+\frac{rx^{*3}}{K}\left|z_1(t)\right|\left|z_1(s)\right|\right. \\
& \qquad\qquad \left.+\frac{2rx^{*3}}{K}\left|z_1(t)\right|\left|z_1(s-\tau)\right|+cx^*y^*\left|z_1(t)\right|\left|z_2(s)\right|\right]ds \\
& +\frac{Mr\left(1+x^{*2}\right)}{K\left(1+m^2\right)^2}\int_{t-\tau}^{t}\left[rx^{*2}\left|z_1(t)\right|\left|z_1(s)\right|+\frac{rx^{*3}}{K}\left|z_1(t)\right|\left|z_1(s)\right|\right. \\
& \qquad\qquad \left.+\frac{2rx^{*3}}{K}\left|z_1(t)\right|\left|z_1(s-\tau)\right|+cx^*y^*\left|z_1(t)\right|\left|z_2(s)\right|\right]ds \\
& +\frac{M^3r^2x^*\left(1+x^{*2}\right)}{K^2\left(1+m^2\right)^2}\int_{t-\tau}^{t}\left|z_1(t)\right|\left|z_1(s-\tau)\right|ds \\
& +\frac{Mr^2x^{*3}\left(1+x^{*2}\right)}{K^2\left(1+m^2\right)^2}\int_{t-\tau}^{t}\left|z_1(t)\right|\left|z_1(s-\tau)\right|ds \\
& +\frac{2Mr^2x^{*3}\left(1+x^{*2}\right)}{K^2\left(1+m^2\right)^2}\int_{t-\tau}^{t}\left\{\left|z_1(t)\right|\left|z_1(s-\tau)\right|\left[1+z_1(s)\right]\right. \\
& \qquad\qquad \left.+\left|z_1(t)\right|\left|z_1(s-\tau)\right|\right\}ds
\end{aligned}
$$

$$\leq -\left[\frac{rx^*\left(1+3x^{*2}\right)}{K\left(1+M^2\right)}+\frac{cx^*y^*}{2\left(1+m^2\right)}+\frac{cx^*y^*}{1+M^2}-\frac{rx^*\left(M+3x^*\right)}{1+m^2}\right.$$
$$-\frac{3rx^{*2}\left(M+x^*\right)}{K\left(1+m^2\right)}-\frac{rx^*\left(M+x^*\right)^2}{K\left(1+m^2\right)}-\frac{\delta x^*\left(1+M^2\right)}{2m\left(1+m^2\right)}-\frac{cx^*\left(L+y^*\right)}{1+m^2}$$
$$-\frac{Mr\left(1+x^{*2}\right)\tau}{K\left(1+m^2\right)^2}\left(\frac{3rx^{*2}}{2}+\frac{9rx^{*3}}{2K}+\frac{rx^*}{2K}+\frac{3cx^*y^*}{2}\right)$$
$$\left.-\frac{M^2r\left(1+x^{*2}\right)\tau}{Kx^*\left(1+m^2\right)^2}\left(\frac{rx^{*2}}{2}+\frac{5rx^{*3}}{2K}+\frac{cx^*y^*}{2}\right)-\frac{M^3r^2x^*\left(1+x^{*2}\right)\tau}{2K^2\left(1+m^2\right)^2}\right]z_1^2(t)$$
$$-\left[\frac{\beta y^*}{M}+\frac{cx^*y^*}{2\left(1+m^2\right)}-\frac{\delta x^*\left(1+M^2\right)}{2m\left(1+m^2\right)}\right]z_2^2(t)$$
$$+\frac{Mr\left(1+x^{*2}\right)}{K\left(1+m^2\right)^2}\left[\left(\frac{3rx^{*2}}{2}+\frac{3rx^{*3}}{2K}+\frac{cx^*y^*}{2}\right)\int_{t-\tau}^{t}z_1^2(s)ds\right.$$
$$\left.+cx^*y^*\int_{t-\tau}^{t}z_2^2(s)ds+\left(\frac{rx^*}{2K}+\frac{3rx^{*3}}{K}\right)\int_{t-\tau}^{t}z_1^2(s-\tau)ds\right]$$
$$+\frac{M^2r\left(1+x^{*2}\right)}{Kx^*\left(1+m^2\right)^2}\left[\left(\frac{rx^{*2}}{2}+\frac{rx^{*3}}{2K}\right)\int_{t-\tau}^{t}z_1^2(s)ds\right.$$
$$\left.+\frac{cx^*y^*}{2}\int_{t-\tau}^{t}z_2^2(s)ds+\frac{2rx^{*3}}{K}\int_{t-\tau}^{t}z_1^2(s-\tau)ds\right]$$
$$+\frac{M^3r^2x^*\left(1+x^{*2}\right)}{2K^2\left(1+m^2\right)^2}\int_{t-\tau}^{t}z_1^2(s-\tau)ds \tag{2.64}$$

Let

$$V_2(z(t)) = \frac{Mrx^*\left(1+x^{*2}\right)\left(3Krx^*+3rx^{*2}+Kcy^*+MrK+Mrx^*\right)}{2K^2\left(1+m^2\right)^2}\int_{t-\tau}^{t}\int_{s}^{t}z_1^2(\gamma)d\gamma ds$$
$$+\frac{Mrcy^*\left(1+x^{*2}\right)\left(2x^*+M\right)}{2K\left(1+m^2\right)^2}\int_{t-\tau}^{t}\int_{s}^{t}z_2^2(\gamma)d\gamma ds$$
$$+\frac{Mr^2x^*\left(1+x^{*2}\right)\left(1+6x^{*2}+4Mx^*+M^2\right)}{2K^2\left(1+m^2\right)^2}\int_{t-\tau}^{t}\int_{s}^{t}z_1^2(\gamma-\tau)d\gamma ds$$

(2.65)

then

$$
\begin{aligned}
\dot{V}_2(z(t)) \quad = \quad & \frac{Mrx^*\tau\left(1+x^{*2}\right)\left(3Krx^*+3rx^{*2}+Kcy^*+MrK+Mrx^*\right)}{2K^2\left(1+m^2\right)^2} z_1^2(t) \\
& - \frac{Mrx^*\left(1+x^{*2}\right)\left(3Krx^*+3rx^{*2}+Kcy^*+MrK+Mrx^*\right)}{2K^2\left(1+m^2\right)^2} \int_{t-\tau}^{t} z_1^2(s)ds \\
& + \frac{Mrcy^*\tau\left(1+x^{*2}\right)\left(2x^*+M\right)}{2K\left(1+m^2\right)^2} z_2^2(t) \\
& - \frac{Mrcy^*\left(1+x^{*2}\right)\left(2x^*+M\right)}{2K\left(1+m^2\right)^2} \int_{t-\tau}^{t} z_2^2(s)ds \\
& + \frac{Mr^2x^*\tau\left(1+x^{*2}\right)\left(1+6x^{*2}+4Mx^*+M^2\right)}{2K^2\left(1+m^2\right)^2} z_1^2(t-\tau) \\
& - \frac{Mr^2x^*\left(1+x^{*2}\right)\left(1+6x^{*2}+4Mx^*+M^2\right)}{2K^2\left(1+m^2\right)^2} \int_{t-\tau}^{t} z_1^2(s-\tau)ds
\end{aligned}
\qquad (2.66)
$$

and then we have from (2.64) and (2.66) that for $t \geq \widehat{T}$

$$
\begin{aligned}
\dot{V}_1(z(t))+\dot{V}_2(z(t)) \quad \leq \quad & -\left[\frac{rx^*\left(1+3x^{*2}\right)}{K(1+M^2)} + \frac{cx^*y^*}{2(1+m^2)} + \frac{cx^*y^*}{1+M^2}\right. \\
& - \frac{rx^*(M+3x^*)}{1+m^2} - \frac{3rx^{*2}(M+x^*)}{K(1+m^2)} - \frac{rx^*(M+x^*)^2}{K(1+m^2)} \\
& - \frac{\delta x^*\left(1+M^2\right)}{2m(1+m^2)} - \frac{cx^*(L+y^*)}{1+m^2} \\
& - \frac{Mrx^*\tau\left(1+x^{*2}\right)\left(3Krx^*+3rx^{*2}+Kcy^*+MrK+Mrx^*\right)}{2K^2\left(1+m^2\right)^2} \\
& - \frac{Mr\left(1+x^{*2}\right)\tau}{K\left(1+m^2\right)^2}\left(\frac{3rx^{*2}}{2} + \frac{9rx^{*3}}{2K} + \frac{rx^*}{2K} + \frac{3cx^*y^*}{2}\right) \\
& - \frac{M^2r\left(1+x^{*2}\right)\tau}{Kx^*\left(1+m^2\right)^2}\left(\frac{rx^{*2}}{2} + \frac{5rx^{*3}}{2K} + \frac{cx^*y^*}{2}\right) \\
& \left. - \frac{M^3r^2x^*(1+x^{*2})\tau}{2K^2\left(1+m^2\right)^2}\right] z_1^2(t)
\end{aligned}
$$

$$-\left[\frac{\beta y^*}{M}+\frac{cx^*y^*}{2(1+m^2)}-\frac{\delta x^*(1+M^2)}{2m(1+m^2)}-\frac{Mrcy^*\tau(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2}\right]z_2^2(t)$$
$$+\frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}z_1^2(t-\tau) \tag{2.67}$$

Let

$$V_3(z(t)) = \frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}\int_{t-\tau}^{t} z_1^2(s)ds \tag{2.68}$$

then

$$\dot{V}_3(z(t)) = \frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}z_1^2(t) - \frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}z_1^2(t-\tau) \tag{2.69}$$

Now define a Lyapunov functional $V(z(t))$ as

$$V(z(t)) = V_1(z(t))+V_2(z(t))+V_3(z(t)) \tag{2.70}$$

then we have from (2.67) and (2.69) that for $t \geq \widehat{T}$

$$\begin{aligned}\dot{V}(z(t)) \leq & -\left[\frac{rx^*(1+3x^{*2})}{K(1+M^2)}+\frac{cx^*y^*}{1+M^2}+\frac{cx^*y^*}{2(1+m^2)}-\frac{rx^*(M+3x^*)}{1+m^2}\right.\\ & -\frac{3rx^{*2}(M+x^*)}{K(1+m^2)}-\frac{rx^*(M+x^*)^2}{K(1+m^2)}-\frac{\delta x^*(1+M^2)}{2m(1+m^2)}-\frac{cx^*(L+y^*)}{1+m^2}\\ & -\frac{Mr^2x^*\tau(1+x^{*2})(3Kx^*+9x^{*2}+MK+5Mx^*+1+M^2)}{K^2(1+m^2)^2}\\ & \left.-\frac{Mrcy^*\tau(1+x^{*2})(4x^*+M)}{2K(1+m^2)^2}\right]z_1^2(t)\\ & -\left[\frac{\beta y^*}{M}+\frac{cx^*y^*}{2(1+m^2)}-\frac{\delta x^*(1+M^2)}{2m(1+m^2)}-\frac{Mrcy^*\tau(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2}\right]z_2^2(t)\\ \equiv & -\zeta z_1^2(t)-\eta z_2^2(t)\end{aligned} \tag{2.71}$$

Then it follows from (2.57) and (2.58) that $\zeta > 0$ and $\eta > 0$. Let $w(s) = \widehat{N}s^2$ where $\widehat{N} = \min\{\zeta, \eta\}$, then w is nonnegative continuous on $[0, \infty)$, $w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (2.71) that for $t \geq \widehat{T}$

$$\dot{V}(z(t)) \leq -\widehat{N}\left[z_1^2(t) + z_2^2(t)\right] = -\widehat{N}\|z(t)\|^2 = -w(\|z(t)\|) \tag{2.72}$$

Now, we want to find a function u such that $V(z(t)) \geq u(\|z(t)\|)$. It follows from (2.61), (2.65), and (2.68) that

$$V(z(t)) \geq \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \tag{2.73}$$

then by Theorem 2.1, we have that $u(s) = \widetilde{N}s^2$ and $V(z(t)) \geq u(\|z(t)\|)$. So the equilibrium point E^* of the system (2.1) is globally asymptotically stable with $p(x) = \frac{cx^2}{1+x^2}$

Remark *2.1* *By Theorem 2.1, Theorem 2.2, and Theorem 2.3 we can assume that*

$$\begin{aligned} V(z(t)) = {} & \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \\ & + a(p(x))\frac{Mrx^*}{2K^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma)d\gamma ds \\ & + b(p(x))\frac{Mr^2x^*}{2K^2}\left[\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma - \tau)d\gamma ds + \tau\int_{t-\tau}^{t} z_1^2(s)ds\right] \\ & + c(p(x))\frac{Mrcy^*}{2K}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds \end{aligned}$$

Then

(i) for $p(x) = cx$ in the Holling-Type I model

$$a(p(x)) = 0 \quad , \quad b(p(x)) = 1 \quad , \quad c(p(x)) = 1$$

(ii) for $p(x) = \frac{cx}{1+x}$ in the Holling-Type II model

$$\begin{aligned} a(p(x)) &= \frac{r(1 + x^*)}{(1 + m)^2} \\ b(p(x)) &= \frac{(1 + x^*)(M + 1 + 2x^*)}{(1 + m)^2} \\ c(p(x)) &= \frac{1 + x^*}{(1 + m)^2} \end{aligned}$$

(iii) for $p(x) = \frac{cx^2}{1+x^2}$ *in the Holling-Type III model*

$$
\begin{aligned}
a(p(x)) &= \frac{\left(1+x^{*2}\right)\left(3Krx^{*}+3rx^{*2}+Kcy^{*}+MKr+Mrx^{*}\right)}{\left(1+m^{2}\right)^{2}} \\
b(p(x)) &= \frac{\left(1+x^{*2}\right)\left(1+6x^{*2}+4Mx^{*}+M^{2}\right)}{\left(1+m^{2}\right)^{2}} \\
c(p(x)) &= \frac{\left(1+x^{*2}\right)\left(2x^{*}+M\right)}{\left(1+m^{2}\right)^{2}}
\end{aligned}
$$

3 Examples

Example 3.1 *In the Holling-Type I we consider the system*

$$
\begin{aligned}
\dot{x}(t) &= x(t)\left[3-10x(t-\tau)-15y(t)\right] \\
\dot{y}(t) &= y(t)\left[1-6\frac{y(t)}{x(t)}\right]
\end{aligned}
\tag{3.1}
$$

where $r = 3$, $K = \frac{3}{10}$, $c = 15$, $\delta = 1$, $\beta = 6$, and $E^* = (\frac{6}{25}, \frac{1}{25})$. Then

$$
\begin{aligned}
& r - cL = 2.24246 > 0 \\
& \delta x^* - cy^*M = 0.0582 > 0 \\
& \frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mr^2x^*\tau}{K^2} - \frac{Mrcy^*\tau}{2K} = 1.5940 > 0 \\
& \frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mrcy^*\tau}{2K} = 0.0103 > 0
\end{aligned}
$$

whenever $\tau = \frac{1}{300}$. Consequently, by Theorem 2.1, we conclude that the unique positive equilibrium point E^* of the system (3.1) is globally asymptotically stable. The trajectory of the system (3.1) is dipicted in Figure 3.1.

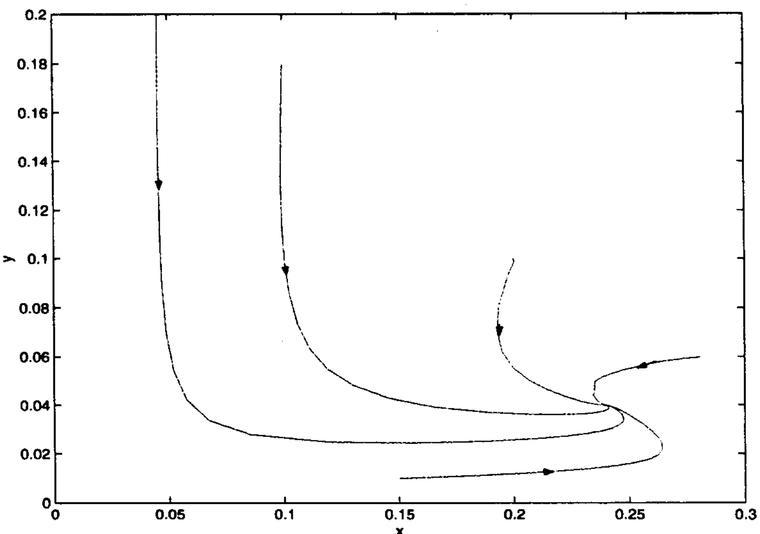

Figure 3.1: The trajectory of the system (3.1) with $\tau = \frac{1}{300}$.

Example 3.2 *In the Holling-Type II we consider the system*

$$\begin{aligned} \dot{x}(t) &= x(t)\left[3 - 10x(t-\tau) - 15\frac{y(t)}{1+x(t)}\right] \\ \dot{y}(t) &= y(t)\left[1 - 6\frac{y(t)}{x(t)}\right] \end{aligned} \tag{3.2}$$

where $r = 3$, $K = \frac{3}{10}$, $c = 15$, $\delta = 1$, $\beta = 6$, and $E^* = (\frac{1}{4}, \frac{1}{24})$. Then

$$\begin{aligned} & r - cL = 2.24246 > 0 \\ & \delta x^*(1+m) - cy^*M = 0.08843 > 0 \\ & \frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{rx^*(x^*+M)}{K(1+m)} \\ & -\frac{Mr^2x^*\tau(1+x^*)(K+3x^*+1+M)}{K^2(1+m)^2} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2} = 0.05316 > 0 \\ & \frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2} = 0.00475 > 0 \end{aligned}$$

whenever $\tau = \frac{1}{300}$. Consequently, by Theorem 2.2, we conclude that the unique positive equilibrium point E^* of the system (3.2) is globally asymptotically stable. The trajectory of the system (3.2) is dipicted in Figure 3.2.

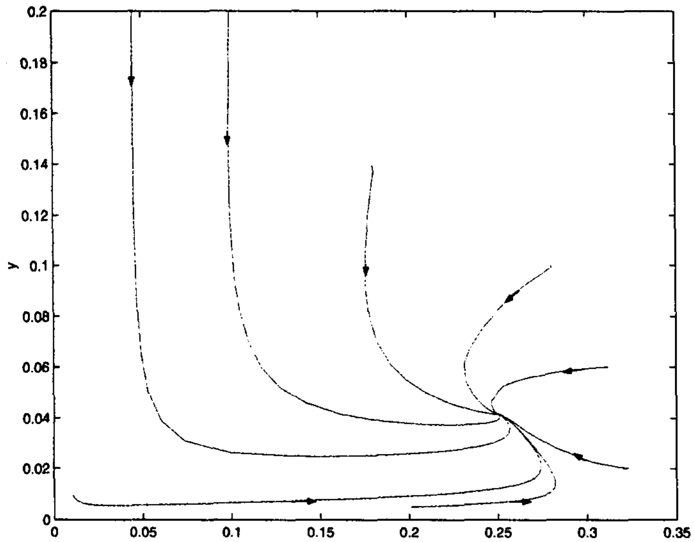

Figure 3.2: The trajectory of the system (3.2) with $\tau = \frac{1}{300}$.

4 Conclusion

In this paper, we obtain that the sufficient condition for the global stability of the Leslie-Gower predator-prey system in Holling-Type I, Holling-Type II, and Holling-Type III models with time delay, respectively. But we believe that the global stability of the predator-prey model with time delay with all different functional response of the predator, $p(x)$, for instance, $p(x) = mx$, $p(x) = \frac{mx}{a+x}$, $p(x) = \frac{mx^2}{a+x^2}$, $p(x) = mx^c$, $0 < c \leq 1$, or $p(x) = m(1 - e^{-cx})$ will be an important topic for future study.

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具時滯參數且函數反應為 Holling 型式的 Leslie-Gower 捕食系統之整體穩定性

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摘　要

本篇討論文主要探討具時滯參數且函數反應為Holling 型式的Leslie-Gower 捕食系統之整體穩定性。首先，我們利用 Lyapunov 函數來分別討論不同 Holling 型式時具時滯參數的 Leslie-Gower 捕食系統之整體穩定性。最後，我們舉例說明我們的結果。

關鍵詞：Leslie-Gower 捕食系統，整體穩定性，時滯參數，Holling 型式，Lyapunov 函數。

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